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On the Skew Laplacian Energy of a Digraph

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Abstract

In this paper we introduce the concept of the skew Laplacian energy of a simple, connected digraph. We derive an explicit formula for the skew Laplacian energy of a digraph G. We also find the minimal value of this energy in the class of all connected digraphs on $n \geq 2$ vertices.

Mathematics Subject Classification: 05C50

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1 INTRODUCTION

Let G be a simple (n, m) digraph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and arc set $\Gamma(G) \subset V(G) \times V(G)$. The skew-adjacency matrix of G is the $n \times n$ matrix $S(G) = [a_{ij}]$ where $a_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(G)$, $a_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(G)$, $a_{ij} = 0$ otherwise. Hence $S(G)$ is a skew symmetric matrix of order n and all its eigen values are of the form $i\lambda$ where $i = \sqrt{-1}$ and $\lambda \in R$. The skew energy of G is sum of the absolute values of eigenvalues of $S(G)$. For additional informations on skew energy of digraphs we refer to [1]. Let $D(G) = diag(d_1, d_2, d_3, ..., d_n)$ the digonal matrix with the vertex degrees $d_1, d_2, d_3, ..., d_n$ of $v_1, v_2, ..., v_n$. Then $L(G) = D(G) - S(G)$ is called the Laplacian matrix of the digraph G. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be the eigenvalues of $L(G)$. Then the set $\sigma_{SL}(G) = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n\}$ is called the skew Laplacian spectrum of the digraph G. The Laplacian matrix of a simple, undirected graph G is $D(G) - A(G)$ where $A(G)$ is the adjacency matrix of G. It is symmetric, singular, positive semi-definite and all its eigenvalues are real and non negative. It is well known that the smallest eigen value is zero and its multiplicity is equal to the number of connected components of G. The Laplacian spectrum of the undirected graph G is the sum of squares of eigenvalues of its Laplacian matrix. For the results and background on the Laplacian spectrum we refer to [2],[3] and the references contained therein.

In this paper we will consider the problem of deriving a formula for $E_{SL}(G)$, the skew Laplacian energy of a digraph G interms of degrees of its vertices. A similar problem for the usual Laplacian energy has been considered in [4]. We also find the minimal value of $E_{SL}(G)$, in the class of all connected digraphs on $n \geq 2$ vertices.

2 FORMULA AND BOUNDS FOR THE SKEW LAPLACIAN ENERGY OF A DIGRAPH

We begin by giving the formal definition of skew Laplacian energy.

Definition 2.1 Let $S(G)$ be the skew adjacency matrix of a simple digraph G. Then the skew Laplacian energy of the digraph G is defined as

$$E_{SL}(G) = \sum_{i=1}^{n} \lambda_i^2$$

where n is the order of G and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ are the eigenvalues of the Laplacian matrix $L(G) = D(G) - S(G)$ of the digraph G.

Example 2.2 Let G be a directed path on four vertices with the arc set (1,2)(2,3)(4,3).

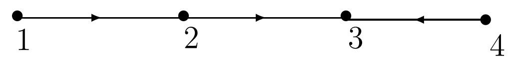

Fig.1 G $= P_4$

Then

$$S(G) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad D(G) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and so

$$L(G) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The eigenvalues of $L(G)$ are $\frac{3}{2}+\frac{1}{2}i+\frac{1}{2}\sqrt{-4+2i}$, $\frac{3}{2}-\frac{1}{2}i+\frac{1}{2}\sqrt{-4-2i}$, $\frac{3}{2}+\frac{1}{2}i-\frac{1}{2}\sqrt{-4+2i}$, $\frac{3}{2}-\frac{1}{2}i-\frac{1}{2}\sqrt{-4-2i}$, and the skew Laplacian energy of the G is $E_{SL}(G) = 4$.

Example 2.3 Let G be a directed cycle on four vertices with the arc set

$\{(1,2),(2,3),(3,4),(4,1)\}$.

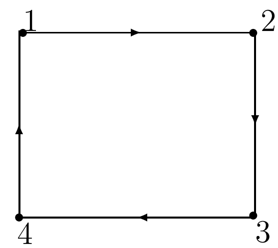

Fig.2. G = C_4

Then

$$L(G) = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

The eigenvalues of $L(G)$ are $2 + i\sqrt{2}, 2 + i\sqrt{2}, 2 - i\sqrt{2}, 2 - i\sqrt{2}$, and the skew Laplacian energy of the G is $E_{SL}(G) = 8$.
Let $G_1 = (V(G_1), \Gamma(G_1))$ and $G_2 = (V(G_2), \Gamma(G_2))$ be two finite, simple directed graphs with disjoint sets of vertices $V(G_1)$ and $V(G_2)$. Then the direct sum $G = G_1 \dot{+} G_2$ of these graphs is defined by $V(G) = V(G_1) \cup V(G_2)$ and $\Gamma(G) = \Gamma(G_1) \cup \Gamma(G_2)$. Then we have the following theorem.

Theorem 2.4. If $G = G_1 \dot{+} G_2$ is the direct sum of two finite, simple digraphs G_1, G_2, then

$$\sigma_{SL}(G) = \sigma_{SL}(G_1) \cup \sigma_{SL}(G_2).$$

By Theorem 2.4 we immediately get the following theorem.

Theorem 2.5. If G is a disconnected digraph whose components are $G_1, G_2, ..., G_m$, Then

$$E_{SL}(G) = \sum_{i=1}^{m} E_{SL}(G_i).$$

Theorem 2.6. Let G be a simple digraph with vertex degrees are $d_1, d_2, ..., d_n$. Then we have

$$E_{SL}(G) = \sum_{i=1}^{n} d_i(d_i - 1). \tag{2.1}$$

Proof. Let G be a simple digraph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and $d(v_i) = d_i$ for $i = 1, 2, ..., n$. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be the eigenvalues of the Laplacian matrix $L(G) = D(G) - S(G)$ where $D(G) = diag(d_1, d_2, ..., d_n)$ and $S(G)$ is the skew-adjacency matrix of digraph G. We have $\sum_{i=1}^{n} \lambda_i =$ sum of determinants of all 1×1 principal submatrices of $L(G) =$ trace of $L(G)$ $= \sum_{i=1}^{n} d_i$. Note that $\sum_{i<j} \lambda_i\lambda_j =$ sum of determinants of all 2×2 principal submatrices of $L(G)$

$$= \sum_{i<j} det \begin{pmatrix} d_i & -a_{ij} \\ -a_{ji} & d_j \end{pmatrix} = \sum_{i<j} (d_id_j - a_{ij}a_{ji}) = \sum_{i<j} \left(d_id_j + {a_{ij}}^2\right).$$

Thus we have

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} d_i$$

and

$$\sum_{i<j} \lambda_i\lambda_j = \sum_{i<j} d_id_j + \sum_{i<j} {a_{ij}}^2.$$

Since ${a_{ij}}^2 = |a_{ij}|$ for every $i < j$, we find that

$$\sum_{i \neq j} \lambda_i\lambda_j = 2\sum_{i<j} \lambda_i\lambda_j = \sum_{i \neq j} d_id_j + \sum_{i \neq j} |a_{ij}| = \sum_{i \neq j} d_id_j + \sum_{i=1}^{n} d_i.$$

Therefore

$$E_{SL}(G) = \sum_{i=1}^{n} \lambda_i^2 = \left(\sum_{i=1}^{n} \lambda_i\right)^2 - \sum_{i \neq j} \lambda_i\lambda_j$$

$$= \left(\sum_{i=1}^{n} d_i\right)^2 - \left[\sum_{i \neq j} d_id_j + \sum_{i=1}^{n} d_i\right]$$

$$= \sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n} d_i$$

$$= \sum_{i=1}^{n} d_i(d_i - 1).$$

This completes the proof.

We remark that Theorem 2.6 shows that the skew Laplacian energy of a digraph is independnt of its orientation.

Corollary 2.7. For any simple, digraph G, its skew Laplacian energy $E_{SL}(G)$ is an even integer.

Proof. Since $d_i \geq 1$ for $i = 1, 2, ..., n$, we have $d_i(d_i - 1)$ is even for $i = 1, 2, ..., n$. Hence their sum $\sum_{i=1}^{n} d_i(d_i - 1)$ is also an even integer.

Corollary 2.8 If H is a proper subgraph of a connected digraph G with atleast three vertices, then $E_{SL}(H) < E_{SL}(G)$

Proof. Since G has atleast one vertex of degree more than one, we find that $E_{SL}(G) > 0$. Since H is obtained by removing atleast one edge from the connected digraph G, using (2.1) we conclude that $E_{SL}(H) < E_{SL}(G)$.

Corollary 2.9. The skew Laplacian energy of a directed path P_n with $n \geq 2$ vertices is $2n - 4$.

Proof. Since every path P_n $(n \geq 2)$ has exactly two vertices of degree one and $(n-2)$ vertices of degree 2, using (2.1) we conclude that $E_{SL}(P_n) = 2(n-2)$.

Corollary 2.10. The skew Laplacian energy of a directed cycle C_n with $n(\geq 3)$ vertices is $2n$.

Proof. Since every vertex in C_n has degree two, it follows from (2.1) that

$$E_{SL}(C_n) = \sum_{i=1}^{n} d_i(d_i - 1) = \sum_{i=1}^{n} 2 = 2n.$$

Theorem 2.11 For any simple, connected digraph on $n \geq 2$ vertices, we have

$$2n-4 \leq E_{SL}(G) \leq n(n-1)(n-2). \tag{2.2}$$

Moreover $E_{SL}(G) = n(n-1)(n-2)$ if and only if G is the complete digraph K_n and $E_{SL}(G) = 2n-4$ if and only if G is the directed path P_n on n vertices.

Proof. Let G be any simple, connected digraph with $n(\geq 2)$ vertices. Since the degree of any vertex is less than or equal to $(n-1)$, we have

$$E_{SL}(G) = \sum_{i=1}^{n} d_i(d_i - 1) \leq \sum_{i=1}^{n}(n-1)(n-2) = n(n-1)(n-2).$$

It is also clear that the maximal skew Laplacian energy in the class of digraphs with n vertices is achieved for the complete digraph K_n. Also only the digraph K_n has this maximal energy.

We prove the first inequality by induction. The result is obviously true for $n = 2$. Suppose the result is true for any connected digraph with $n-1$ vertices. We prove the result for any arbitrary connected digraph with n vertices. Let G be any connected digraph with n vertices. Then, there is an induced subgraph $H \subset G$ on $(n-1)$ vertices which is also connected. Denote $V(H) = \{v_1, v_2, ..., v_{n-1}\}$ and $V(G) = V(H) \cup \{v_n\}$. Since G is connected, v_n is adjacent to atleast one vertex $v_i (1 \leq i \leq n-1)$. Assume that v_n is adjacent to v_{n-1}. Denote by G_1 the graph with the same vertex set as G, induced in G by H and the pendent arc $v_{n-1}v_n$. Then we have $H \subset G_1 \subset G$ and $E_{SL}(G_1) \leq E_{SL}(G)$. By the induction hypothisis we have $2n - 6 \leq E_{SL}(H)$. Moreover $E_{SL}(G) = E_{SL}(H) + 2d$ where d is the degree of v_{n-1} in the graph H. Thus

$$E_{SL}(G) \geq E_{SL}(G_1) = E_{SL}(H) + 2d \geq (2n-6) + 2 = 2n - 4.$$

Hence the first inequality in (2.2) is also true for G. By Corollary 2.9 we have $E_{SL}(P_n) = 2n - 4$. We will now prove that if G is a simple, connected digraph with $n \geq 2$ vertices such that $E_{SL}(G) = 2n - 4$, then G must be a directed path P_n. Again we prove this by induction on n. Let H and G_1 have the same meaning as in the previous part of the proof. We prove that $d_n = d(v_n) = 1$. On the contrary, let us assume that $d_n \geq 2$ and let v_n be adjacent not only to v_{n-1} but also to some $v_{n-2} \in V(H)$. We have

$$E_{SL}(H) \geq 2n - 6$$

and

$$E_{SL}(G) \geq E_{SL}(H) + 2d_H(v_{n-2}) + 2d_H(v_{n-1}) + 2.$$

Thus

$$2n - 4 = E_{SL}(G) \geq (2n-6) + 2 + 2 + 2 = 2n,$$

which is a contradiction. Hence $d_n = d(v_n) = 1$, so G is the same as G_1. Since

$$E_{SL}(G) = E_{SL}(G_1) = E_{SL}(H) + 2d_H(v_{n-1}) = 2n - 4,$$

we obtain

$$2d_H(v_{n-1}) = (2n-4) - E_{SL}H \leq (2n-4) - (2n-6) = 2.$$

This implies

$$d_H(v_{n-1}) = 1.$$

Therefore

$$E_{SL}(H) = (2n-4) - 2 = 2n - 6.$$

By the induction hypothesis this means that H is the path P_{n-1} and v_{n-1} is end vertex of this path. This yields that G must be the path P_n. This completes the proof.

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References

[1] C. Adiga, R. Balakrishnan and Wasin So, The skew energy of a graph (communicated for publication).

[2] R. Grone, R. Merris, V.Sunder, The Laplacian spectrum of a graph, SIAM J.Matrix Anal. Appl. 11(1990), 218-238.

[3] R. Merris, Laplacian matrices of graphs, A survey, Linear Algebra and its Appl. 197,198 (1994), 143-176.

[4] Mirjana Lazic Kragujevac, On the Laplacian energy of a graph, Czechoslovak Math.Jour. 56(131) (2006), 1207-1213.

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