International Mathematical Forum, 4, 2009, no. 31, 1511 - 1514

On the Chromatic Number of Some Harary Graphs

H. Abdollahzadeh Ahangar*

*Department of Mathematics
University of Mysore, Manasagangotri
Mysore- 570006, India
ha.ahangar@yahoo.com

L. Pushpalatha†

†Departmant of Mathematics
Yuvaraja's College
Mysore-570005, India
pushpakrishna@yahoo.com

Abstract

In this paper we answer a question posed [1]. We show that $\chi(H_{2m,3m+2}) = m + k + 1$ such that $k = \lceil m/2 \rceil$ and $H_{2m,3m+2}$ is the Harary graph.

Mathematics Subject Classification: 05C15

Keywords: Chromatic number, Harary's graph

1 Introduction

Throughout this paper, all graphs considered are finite, undirected, loopless and without multiple edges. We refer the reader to [2] for terminology in graph theory. A k-coloring of a graph G is a labeling $f : V(G) \to T$, where $|T| = k$ and it is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G)$ is the least k such that G is k-colorable [2].

Definition [2]. The Harary graph $H_{k,n}$ is defined as follows:
Case 1. k even. Let $k = 2r$, then $H_{2r,n}$ is constructed as follows:
It has vertices 0,1,...,$n-1$ and two vertices i and j are joined if $i-r \le j \le i+r$ (where addition is taken modulo n).
Case 2. k odd and n even. Let $k = 2r + 1$, then the $H_{2r+1,n}$ is constructed by

first drawing $H_{2r,n}$ and then adding edges joining vertex i to vertex $i+(n/2)$ for $1 \leq i \leq n/2$.
Case 3. k odd and n odd. Let $k = 2r+1$, then $H_{2r+1,n}$ is constructed by first drawing $H_{2r,n}$ and then adding edges joining vertex 0 to vertices $(n-1)/2$ and $(n+1)/2$ and vertex i to vertex $i+(n+1)/2$ for $1 \leq i \leq (n-1)/2$.

2 $\chi(H_{2m,3m+2}) \neq m+2$ for $m \geq 3$

We recall here the well known question which first appeared in print in [1].

Question. [1] If $m \geq 3$, then can one say $\chi(H_{2m,3m+2}) \neq m+2$?

Lemma 2.1 $\chi(H_{2m,3m+2}) = 6$ *for* $m = 3$.

Proof. Let $G = H_{6,11}$ and its vertices be $v_1, v_2, ..., v_{11}$. The subgraph induced by the vertex set $\{v_1, ..., v_4\}$ is a complete graph K_4, thus we are forced to use four colors on its vertices. Suppose that the colors 1, 2, 3 and 4 are colors that appear on the induced subgraph K_4.
Also, the induced subgraph on the vertices v_7,...,v_{10} is a complete graph K_4, thus we can use four used colors 1, 2, 3 and 4 on vertices of the new induced subgraph respectively.
Since the vertex v_5 is adjacent to the vertices v_2, v_3, v_4, v_7 and v_8 and used colors 1, 2, 3, 4 on these vertices then it is forced to take a new color 5. Also the vertex v_6 is adjacent to vertices which used colors 1, 2, 3, 4, 5 on those vertices, then v_6 is forced to take a new color 6. Finally, for the end vertex v_{11} by an argument similar to that described for the two vertices v_5 and v_6 settles for the vertex v_{11}. So it can take one of the colors 5 or 6. This implies that $\chi(H_{6,11}) = 6$.

We are now ready to prove the main result of this section.

Theorem 2.2 $\chi(H_{2m,3m+2}) = m+k+1$ *where* $k = \lceil m/2 \rceil$ *and* $m \geq 3$.

Proof. Let $G = H_{2m,3m+2}$ for $m \geq 3$ and its vertices be $v_1, v_2, ..., v_{3m+2}$. Since the cardinality of the vertex set is $3m+2$, we have two induced complete subgraphs K_{m+1} with vertex sets $\{v_1, v_2, ..., v_{m+1}\}$ and $\{v_{m+k+2}, v_{m+k+3}, v_{m+k+4}, ..., v_{2(m+1)+k}\}$, respectively. Therefore we must use $m+1$ colors for those vertices. We can assume that we use colors 1, 2, ..., $m+1$ for the first induced complete subgraph stated. Also, we can use $m+1$ used colors on the first induced complete subgraph for the second induced complete subgraph, respectively. Then we have used only $m+1$ colors for $2m+2$ stated vertices from the

graph G. If we color the remaining m vertices, then our proof will be complete. The vertex v_{m+2} cannot take a color from set $\{1, 2, ..., m+1\}$, because it is adjacent to $m+1$ vertices from set $\{v_2, v_3, ..., v_{m+1}, v_{m+k+2}\}$ and we have used colors 2, ..., $m+1$, 1 respectively, for those vertices. Thus we are forced to use a new color, for example $m+2$ for the vertex v_{m+2}. Also the vertex v_{m+3} cannot take a color from set $\{1, 2, ..., m+1, m+2\}$, because it is adjacent to $m+2$ vertices from set $\{v_3, v_4, ..., v_{m+2}, v_{m+k+2}, v_{m+k+3}\}$ and we have used colors 3, 4, ..., $m+2$, 1, 2 respectively, for those vertices. Thus we are forced to use a new color for example $m+3$ for the vertex v_{m+3}. An argument similar to that described above for two vertices v_{m+2} and v_{m+3} settles our proof for the vertices v_{m+4},...,v_{m+k+1}. Then we can use colors $m+4$, $m+5$,..., $m+k+1$ for vertices v_{m+4},...,v_{m+k+1}, respectively. We color the remaining vertices v_{2m+k+3}, v_{2m+k+4},..., v_{3m+2} as follows:
Since the vertex v_{2m+k+3} is nonadjacent to vertex v_{m+2} then it can be colored $m+2$. Also, the vertex v_{2m+k+4} is nonadjacent to vertex v_{m+3} then it can be colored $m+3$. An argument similar to that described above for two vertices v_{2m+k+3} and v_{2m+k+4} settles our proof for the vertices v_{2m+k+5},...,v_{3m+2}. This implies that $\chi(H_{2m,3m+2}) = m+k+1$ where $k = \lceil m/2 \rceil$ and $m \geq 3$.

In the last part we give the following example to illustrate the Theorem above.

Example 2.3 *Chromatic number for the graph $H_{10,17}$ is 9.*

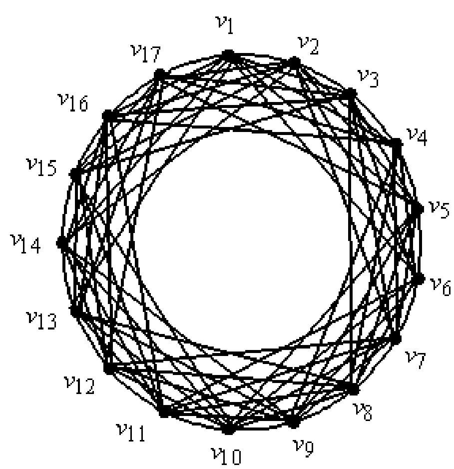

Figure 1.

Let $G = H_{10,17}$, and c be a proper coloring for it. We obtain a c-coloring for it as follows:
$c(v_1) = c(v_{10}) = 1$, $c(v_2) = c(v_{11}) = 2$, $c(v_3) = c(v_{12}) = 3$, $c(v_4) = c(v_{13}) = 4$,

$c(v_5) = c(v_{14}) = 5$, $c(v_6) = c(v_{15}) = 6$, $c(v_7) = 7$, $c(v_8) = 8$, $c(v_9) = 9$, $c(v_{16}) = 7$, $c(v_{17}) = 8$. This implies that $\chi(H_{10,17}) = 9$.

References

[1] . P. Kazemi, Chromatic Number in Some Graphs. International Mathematical Forum, 2. 2007, no, 35, (1723-1727).

[2] . B. West, Introduction to graph theory. Prentice Hall of India, (2003).

Received: November, 2008