A generalization of chromatic index

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Abstract
Let $G = (V, E)$ be a graph and $k \geq 2$ an integer. The general chromatic index $\chi'_k(G)$ of $G$ is the minimum order of a partition $P$ of $E$ such that for any set $F$ in $P$ every component in the subgraph $\langle F \rangle$ induced by $F$ has size at most $k - 1$. This paper initiates a study of $\chi'_k(G)$ and generalizes some known results on chromatic index.

The purpose of this paper is to obtain a generalization of chromatic index. Compared to many generalizations of chromatic number, there exist very few generalizations of chromatic index in the literature. For example, see [2] and [3].

Let $G = (V, E)$ be a graph and $k \geq 2$ an integer. A set $F \subseteq E$ is a $k$-set (or $k$-independent set) if every component in the subgraph $\langle F \rangle$ induced by $F$ has size at most $k - 1$. Equivalently, a set $F \subseteq E$ is $k$-independent if the sum of the degrees of the vertices in every component of the subgraph $\langle F \rangle$ is $r$, where $2 \leq r \leq 2(k - 1)$.

A partition $\{E_1, E_2, \ldots, E_r\}$ of $E$ is an $I_k$-partition if each $E_i$ is an $I_k$-set. An $I_k$-edge coloring of $G$ is a coloring of the edges of $G$ so that the set of all edges receiving the same color is an $I_k$-set. An $I_k$-edge coloring which uses $r$ colors is called a $(k, r)$-edge coloring.

The $k$-chromatic index $\chi'_k = \chi'_k(G)$ of $G$ is the minimum number of colors needed in an $I_k$-edge coloring of $G$. If $\chi'_k(G) = n$, then $G$ is said to be $(k, n)$-edge chromatic. The $k$-edge independence number $\beta_{1k} = \beta_{1k}(G)$ of $G$ is the maximum cardinality of an $I_k$-set. Clearly, if $M$ is any independent set of edges, then $M$ is an $I_k$-set for all $k \geq 2$.

We observe that $\chi'_2(G) = \chi(G)$, the chromatic index. Also $\beta_{12} = \beta_1$, the edge independence number of $G$. If $G$ has size $q$, then $\chi'_k(G) = 1$ for all $k > q$. If $L(G)$ is the line graph of $G$, then

$$\chi(G) = \chi(L(G))$$

(1)

where $\chi(L(G))$ is the chromatic number of $L(G)$.

Correspondence to: E. Sampathkumar, Department of Mathematics, Mysore University, Mysore 570006, India.
The vertex analogue of $\chi_k(G)$ has been defined by Sampathkumar [5] as follows: Let $k \geq 2$ be an integer. The $k$-chromatic number $\chi_k(G)$ of $G$ is the minimum order of a partition $\{V_1, V_2, \ldots, V_k\}$ of $V$ such that every component in the subgraph $\langle V_i \rangle$ induced by $V_i$ has order at most $k-1$. Clearly, for any graph $G$ with size $q \geq 1$

$$\chi_k(G) = \chi_k(L(G))$$

(2)

The problem of determining the $k$-chromatic index for the complete graph $K_p$ and the complete bipartite graph $K_{m,n}$ is open. However Figs. 1 and 2 will give the $k$-chromatic index of these graphs in some cases.

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\[ \chi_k'(K_p) \]

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\[ \chi_k'(K_{n,n}) \]

The vertex analogue of $\chi_k'(G)$ has been defined by Sampathkumar [5] as follows: Let $k \geq 2$ be an integer. The $k$-chromatic number $\chi_k'(G)$ of $G$ is the minimum order of a partition $\{V_1, V_2, \ldots, V_k\}$ of $V$ such that every component in the subgraph $\langle V_i \rangle$ induced by $V_i$ has order at most $k-1$. Clearly, for any graph $G$ with size $q \geq 1$.

$$\chi_k'(G) = \chi_k'(L(G))$$

(2)

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\[ \chi_k'(K_p) \]

Also $\chi_{10}(K_7) = 3$, for $k = 10, 11$

$$\chi_k(K_7) = 2,$$ for $11 \leq k \leq 21$

$$\chi_k(K_{8,8}) = 4,$$ for $k = 10, 11$

Also $\chi_k(K_{7,7}) = 3$, for $k = 10, 11$

$$\chi_k(K_8) = 3,$$ for $10 \leq k \leq 14$

$$\chi_k(K_{n,n}) = 3,$$ for $12 \leq k \leq 16$, $n = 7, 8$.

$$\chi_k(K_{9,9}) = 3,$$ for $18 \leq k \leq 28$

$$\chi_k(K_{n,n}) = 2,$$ for $10 \leq k \leq 16$, $4 \leq n \leq 6$.

$$\chi_k(K_{9,9}) = 3,$$ for $28 \leq k \leq 36$

$$\chi_k(K_{n,n}) = 2,$$ for $17 \leq k \leq 25$, $5 \leq n \leq 8$.

$$\chi_k(K_{9,9}) = 2,$$ for $36 \leq k \leq 49$, $6 \leq n \leq 8$.

$$\chi_k(K_{9,9}) = 2,$$ for $49 \leq k \leq 64$.

Let $G$ be a graph of order $p$, and $2 \leq k \leq r$. If $G$ is a cycle, then $\chi_k(G) = 2$. We also observe that for all $2 \leq k \leq r$, an $I_k$-set is an $I_r$-set, and

$$\beta_{12} = \beta_{1k} = \beta_{1r}, \quad (3)$$

$$\chi_1 \leq \chi_k \leq \chi_2 = \chi'. \quad (4)$$
Proposition 1. For any graph $G = (V, E)$, (i) $\beta_{1k} \leq (k-1)\beta_1$, and (ii) $\chi' \leq (k-1)\chi_k$.

**Proof.** (i) Let $F \subseteq E$ be an $I_k$-set with $|F| = \beta_{1k}$. Clearly, the subgraph $\langle F \rangle$ contains at most $\beta_1$ components, and each component containing at most $k-1$ edges. Thus $|F| = \beta_{1k} \leq (k-1)\beta_1$. To establish (ii), let $\{E_1, E_2, \ldots, E_r\}$ be an $I_k$-partition of $E$ with $r = \chi_k(G)$, and $\chi'(\langle E_i \rangle) = t_i$. Then $t_i \leq k-1$ for each $i$, and $\chi'(G) \leq \sum t_i \leq (k-1)\chi_k(G)$.

We now deduce some bounds for $\chi_k$ using (4) and the following results:
If $\Delta$ is the maximum degree of $G$,

$$\Delta \leq \chi' \leq \Delta + 1.$$  \hfill (5)

If $G$ is bipartite

$$\chi = \Delta.$$  \hfill (6)

By (4), (5) and (6), we have for any graph $G$, if $k \geq 2$

$$\left[ \frac{\Delta}{k-1} \right] \leq \chi_k \leq \Delta + 1$$  \hfill (7)

and if $G$ is bipartite,

$$\chi_k \leq \Delta.$$  \hfill (8)

Proposition 2. For any graph $G$ with $q$ edges

(i) $$\frac{q}{\beta_{1k}} \leq \chi_k' \leq \frac{q}{k-1},$$

(ii) $$\frac{q}{(k-1)\beta_1} \leq \chi_k \leq \left[ \frac{q - \beta_{1k}}{k-1} \right] + 1.$$

**Proof.** (i) Let $\{E_1, E_2, \ldots, E_r\}$ be an $I_k$-partition of $E$ with $r = \chi_k$. Then $q = \Sigma |E_i| \leq r\beta_{1k}$, and the lower bound in (i) follows. The upper bound in (i) is trivial. The lower bound in (ii) follows from (i) and (3). To establish the upper bound, let $F \subseteq E$ be an $I_k$-set with $|F| = \beta_{1k}$. Clearly, $\chi_k(G - F) \geq \chi_k - 1$. Since $G - F$ has $q - \beta_{1k}$ edges, we have from (i),

$$\chi_k(G - F) \leq \frac{q - \beta_{1k}}{k-1}.$$  

Therefore,

$$\chi_k(G) \leq \left[ \frac{q - \beta_{1k}}{k-1} \right] + 1.$$  \hfill \(\Box\)
(k, n)-Critical Graphs: Let G be a graph with maximum degree $\Delta$. Then G is chromatic-index critical (or simply, $\Delta$-critical) if (i) G is connected, (ii) $\chi'(G) = \Delta + 1$, and (iii) $\chi'(G - e) < \chi'(G)$ for every edge e of G. For details on $\Delta$-critical graphs, see [1] and [7]. We generalize this concept as follows:

Let $k \geq 2$ and $n \geq 2$ be integers. A graph G is $(k, n)$-critical if (i) G is connected, (ii) $\chi_k(G) = n$, and (iii) $\chi_k(G - e) < \chi_k(G)$ for every edge e of G.

Note that a $\Delta$-critical graph is $(2, \Delta + 1)$-critical. For $k \geq 3$, the star $K_{1,n}$ is $(k, r)$-critical, if and only if, $n \equiv 1 \mod (k - 1)$, where $r = \chi_k(K_{1,n})$. The Peterson graph is $(4, 3)$-critical. This can be seen from the $(4, 3)$-colorings of the edges as in Fig. 3.

Some elementary properties of $(k, n)$-critical graphs are as follows.

**Proposition 3.** Let G be a $(k, n)$-critical graph. If $F \subseteq E$ is an $I_k$-set, then (i) $\chi_k(G - F) = n - 1$, (ii) G contains a $(k, r)$-critical subgraph for every $r$ satisfying $2 < r < n$, and (iii) if $u$ and $v$ are adjacent vertices in G, then $\deg u + \deg v \geq n + 1$.

**Proof.** (i) is trivial.

(ii) For every edge e of G, $\chi_k(G - e) = n - 1$. If the graph $G - e$ is not $(k, n - 1)$-critical, we successively remove the edges from $G - e$ until we obtain a graph $G'$ which is $(k, n - 1)$-critical. Continuing this process, we can obtain a $(k, r)$-critical subgraph of G for each $r, 2 \leq r \leq n$.

(iii) Clearly there exists a $(k, n)$-edge coloring of G such that $\{e\}$ is a color class. Let $\{e\}, E_2, E_3, \ldots, E_n$ be the color classes in such an edge coloring. The edge e should be adjacent to at least one edge in each color class $E_i, 2 \leq i \leq n$. This implies $(\deg u - 1) + (\deg v - 1) \geq n - 1$.

A graph G is $(k, n)$-vertex critical if $\chi_k(G) = n$ and $\chi_k(G - v) = n - 1$ for all $v \in V$. We deduce our next result using a known result.

**Proposition 4** (Sampathkumar [5]). Let G be a $(k, n)$-vertex critical graph, $n \geq 2$. Then (i) G is $(n - 1)$-edge connected, and (ii) $\delta(G) \geq n - 1$, where $\delta(G)$ is the minimum degree of G.
A generalization of chromatic index

Clearly, \( \delta(L(G)) = \min \{ \deg u + \deg v : uv \in E \} - 2 \). Since \( \chi_k(G) = \chi_k(L(G)) \), and \( G \) is \((k, n)\)-critical \( \iff L(G) \) is \((k, n)\)-vertex critical, we deduce the following proposition from Proposition 4:

**Proposition 5.** Let \( G \) be a \((k, n)\)-critical graph, \( n \geq 2 \). Then (i) \( L(G) \) is \((n - 1)\)-edge connected.

**Corollary 5.1.** Let \( G \) be a \( \Delta \)-critical graph. Then \( L(G) \) is \( \Delta \)-edge connected.

We now present an upper bound on the number of edges in a \((k, n)\)-critical graph.

**Proposition 6.** Let \( d_1, d_2, \ldots, d_p \) be the degree sequence of a \((p, q)\) graph \( G \). If \( G \) is \((k, n)\)-critical then \( q \leq \frac{\sum d_i^2}{n + 1} \).

**Proof.** The number of edges in the line graph \( L(G) \) of \( G \) is given by \( q_L = -q + \frac{1}{2} \sum d_i^2 \). Let \( d'_1, d'_2, \ldots, d'_q \) be the degree sequences of \( L(G) \). By (ii) of Proposition 5, \( d'_i \geq \delta(L(G)) \geq n - 1 \) for each \( i \). Hence,

\[
2q_L = \sum_{i=1}^{q} d'_i \geq q(n - 1), \quad \text{and} \quad q \leq \frac{-2q + \sum d_i^2}{n + k - 3}
\]

and the result follows.

**References**

[3] A.J.W. Hilton, Coloring the edges of a multigraph so that each vertex has at most \( j \), or at least \( j \) edges of each color in it, J. London Math. Soc. (2) 12 (1975) 123–128.