A Note on Radius of Starlikeness and Convexity of $p$-Valent Analytic Functions

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Abstract. Let $A_{p}$ denote the class of analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

defined in the unit disc $U = \{z : |z| < 1\}$ and $\Omega$ denote the class of functions such that $\omega(0) = 0$ and $|\omega(z)| < 1$. Let $P(A, B, p, \alpha)$ be the class of functions of the form

$$p(z) = p + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad p(z) = \frac{p + \gamma \omega(z)}{1 + B \omega(z)}, \quad -1 \leq B < A \leq 1$$

where $\gamma = (p - \alpha)A + \alpha B$. In this paper, we define the class $S_q(A, B, p, \alpha)$ of functions $f(z) \in A_p$ such that

$$q + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = p(z) \quad \text{for} \quad p(z) \in P(A, B, p, \alpha)$$
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1. Introduction

Let \( A_p \) denote the class of analytic functions of the form

\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n
\]

defined in the unit disc \( U = \{ z : |z| < 1 \} \). Let \( \Omega \) denote the class of bounded analytic functions \( \omega(z) \) in \( U \) satisfying the conditions \( \omega(0) = 0 \) and \( |\omega(z)| \leq 1 \) (\( z \in U \)).

For functions \( g(z) \) and \( G(z) \) analytic in \( U \), we say that \( g(z) \) is subordinate to \( G(z) \) if there exists a Schwarz function \( \omega(z) \in \Omega \) such that

\[
g(z) = G(\omega(z))
\]

If \( G(z) \) is univalent in \( U \), then \( g(z) \) is subordinate to \( G(z) \) if and only if \( g(0) = G(0) \) and \( g(U) \subseteq G(U) \).

For \( -1 \leq B < A \leq 1 \) and \( 0 \leq \alpha < p \), \( \mathcal{P}(A, B, p, \alpha) \) \(^1\) denote the class of analytic functions defined in \( U \) such that

\[
p(z) = p + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad p(z) = \frac{p + \gamma \omega(z)}{1 + B \omega(z)}, \quad -1 \leq B < A \leq 1
\]

where \( \gamma = (p - \alpha)A + \alpha B \). Further, \( p(z) \in \mathcal{P}(A, B, p, \alpha) \) if and only if

\[
p(z) = (p - \alpha)p_1(z) + \alpha, \quad p_1(z) \in \mathcal{P}(A, B)
\]

where \( \mathcal{P}(A, B) \) \(^3\) is the Janowski class of functions \( p_1(z) \) which are of the form

\[
p_1(z) = 1 + \sum_{n=1}^{\infty} b_n z^n
\]

and are analytic in \( U \), such that \( p_1(z) \in \mathcal{P}(A, B) \) if and only if

\[
p_1(z) = \frac{1 + A \omega(z)}{1 + B \omega(z)}, \quad -1 \leq B < A \leq 1, \quad \omega(z) \in \Omega, \quad z \in U.
\]

We define the class \( S_q(A, B, p, \alpha) \) of functions \( f(z) \in A_p \) such that

\[
q + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = p(z) \quad \text{for} \quad p(z) \in \mathcal{P}(A, B, p, \alpha)
\]
2. MAIN RESULTS

Lemma 2.1. (Jack’s Lemma) [2]: Let $\omega(z)$ be a regular function in the unit disc $U$ with $\omega(0) = 0$, then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_1$, we can write $z_1\omega'(z_1) = k\omega(z_1)$, where $k$ is real and $k \geq 1$.

Lemma 2.2. The function

$$
\omega = \begin{cases} 
\frac{p + \gamma z}{1 + Bz} & \text{for } B \neq 0, \\
p + \gamma z & \text{for } B = 0
\end{cases}
$$

maps $|z| = r$ onto a disc centered at $C(r)$ and having the radius $\rho(r)$ given by

$$
C(r) = \begin{cases} 
\left( \frac{p - \gamma Br^2}{1 - B^2r^2}, 0 \right) & \text{for } B \neq 0, \\
(p, 0) & \text{for } B = 0
\end{cases}
$$

and

$$
\rho(r) = \begin{cases} 
\frac{(\gamma - pB)r}{1 - B^2r^2} & \text{for } B \neq 0, \\
|\gamma|r & \text{for } B = 0.
\end{cases}
$$

Proof. Consider

$$
\begin{cases} 
\omega = \frac{p + \gamma z}{1 + Bz} \Rightarrow z = \frac{\omega - p}{\gamma - B\omega} \Rightarrow |z|^2 = r^2 = \frac{|\omega - p|^2}{|\gamma - B\omega|^2} & \text{for } B \neq 0 \\
\Rightarrow u^2 + v^2 + \left( \frac{2\gamma Br^2 - 2p}{1 - B^2r^2} \right) u + \frac{p^2 - \gamma^2r^2}{1 - B^2r^2} = 0, & \text{for } B \neq 0.
\end{cases}
$$

$$
\begin{cases} 
\omega = 1 + \gamma z \Leftrightarrow z = \frac{\omega - p}{\gamma} \Leftrightarrow |z|^2 = r^2 = \frac{|\omega - p|^2}{|\gamma|^2} & \text{for } B = 0 \\
\Rightarrow u^2 + v^2 - 2u + p^2 - \gamma^2r^2 = 0, & \text{for } B = 0.
\end{cases}
$$

Lemma follows from (2.4).

Lemma 2.3. The function

$$
\omega = \begin{cases} 
\frac{(\gamma - pB)z}{1 + Bz} & \text{for } B \neq 0, \\
\gamma z & \text{for } B = 0
\end{cases}
$$
maps \(|z| = r\) onto a disc centered at \(C(r)\) and having the radius \(\rho(r)\) given by

\[
C(r) = \begin{cases} 
\left(\frac{-B(\gamma - pB)r^2}{1 - B^2r^2}, 0\right) & \text{for } B \neq 0, \\
(0, 0) & \text{for } B = 0
\end{cases}
\]

and

\[
\rho(r) = \begin{cases} 
\frac{(\gamma - pB)r^2}{1 - B^2r^2} & \text{for } B \neq 0, \\
|\gamma| r & \text{for } B = 0.
\end{cases}
\]

**Proof.** Consider

\[
\omega = (\gamma - pB)z 
\iff z = \frac{\omega}{\gamma - pB - B\omega} 
\iff |z|^2 = r^2 = \frac{|\omega|^2}{|\gamma - pB - B\omega|^2} \quad \text{for } B \neq 0
\]

\[
u^2 + v^2 = \left(\frac{2B(\gamma - pB)r^2}{1 - B^2r^2}\right) u + \frac{(\gamma - pB)2r^2}{1 - B^2r^2} = 0, \quad \text{for } B \neq 0.
\]

\[
\omega = \gamma z \iff z = \frac{\omega}{\gamma} \iff |z|^2 = r^2 = \frac{|\omega|^2}{|\gamma|^2} \Rightarrow u^2 + v^2 - \gamma^2 r^2 = 0 \quad \text{for } B = 0.
\]

Lemma follows from (2.8). \(\square\)

**Theorem 2.4.** Let \(f(z) \in \mathcal{A}_p\) be such that

\[
z f^{(q+1)}(z) \quad p + q < \begin{cases} 
(\gamma - pB)z & F_1(z) \quad \text{for } B \neq 0, \\
\gamma z & F_2(z) \quad \text{for } B = 0.
\end{cases}
\]

Then, \(f(z) \in S_q(A, B, p, \alpha)\) and this result is sharp being obtained by the function \(\frac{p + \gamma z}{1 + Bz}\).

**Proof.** Define

\[
\frac{f^{(q)}(z)}{z^{p-q}} = \begin{cases} 
(1 + B\omega(z))^{\frac{\gamma - pB}{\beta}} & \text{for } B \neq 0, \\
e^{\gamma\omega(z)} & \text{for } B = 0
\end{cases}
\]

where \((1 + B\omega(z))^{\frac{\gamma - pB}{\beta}}\) and \(e^{\gamma\omega(z)}\) have the value at 1 at the origin.

Then \(\omega(z)\) is analytic in \(\mathcal{U}\) and \(\omega(0) = 0\). On logarithmic differentiation we
get,
\[
zf^{(q+1)}(z) - p + q < \begin{cases} 
\frac{(\gamma - pB)z\omega'(z)}{1 + B\omega(z)} & \text{for } B \neq 0, \\
\gamma z\omega'(z) & \text{for } B = 0.
\end{cases}
\]

Now it is easy to realize that subordination (2.9) is equivalent to \(|\omega(z)| < 1\) for all \(z \in \mathcal{U}\). By Jack’s Lemma it follows that, there exists a point \(z_1 \in \mathcal{U}\) such that
\[
z_1 f^{(q+1)}(z_1) - p + q < \begin{cases} 
\frac{(\gamma - pB)k\omega(z_1)}{1 + B\omega(z_1)} = F_1(\omega(z_1)) \notin F_1(\mathcal{U}) & \text{for } B \neq 0, \\
\gamma k\omega(z_1) = F_2(\omega(z_1)) \notin F_2(\mathcal{U}) & \text{for } B = 0.
\end{cases}
\]

This contradicts our assumption given by (2.9) and the fact that \(|\omega(z)| < 1\) for all \(z \in \mathcal{U}\).

By using the condition (2.11), we get
\[
zf^{(q+1)}(z) \div f^{(q)}(z) + q = \begin{cases} 
\frac{p + \gamma \omega(z)}{1 + B\omega(z)} & \text{for } B \neq 0, \\
p + \gamma \omega(z) & \text{for } B = 0.
\end{cases}
\]

Now by inequality (2.13) we obtain
\[
zf^{(q+1)}(z) \div f^{(q)}(z) + q < \begin{cases} 
\frac{p + \gamma z}{1 + Bz} & \text{for } B \neq 0, \\
p + \gamma z & \text{for } B = 0.
\end{cases}
\]

By inequality (2.14) it follows that \(f(z) \in S_q(A, B, p, \alpha)\).

**Corollary 2.5.** Let \(f(z) \in S_q(A, B, p, \alpha)\). Then, \(f(z)\) can be written in the form
\[
f^{(q)}_z(z) = \begin{cases} 
z^{p-q}(1 + B\omega(z))^{\frac{\gamma - pB}{B}} & \text{for } B \neq 0, \\
z^{p-q}\gamma \omega(z) & \text{for } B = 0.
\end{cases}
\]

**Theorem 2.6.** The radius of starlikeness and the radius of convexity of the \(S_q(A, B, p, \alpha)\) is given by
\[
R_{sc} = \frac{2(p-q)}{(\gamma - pB) + \sqrt{(\gamma - pB)^2 + 4(p-q)[(\gamma - pB) + (p-q)B^2]}}.
\]
The radius is sharp, being attained by the function

\[
f_s^{(q)}(z) = \begin{cases} 
  z^{p-q}(1 + B\omega(z))^{\frac{\gamma-pB}{B}} & \text{for } B \neq 0, \\
  z^{p-q}e^{\gamma\omega(z)} & \text{for } B = 0.
\end{cases}
\]

(2.17)

\[
f_s^{(q)}(z) = \begin{cases} 
  z^{p-q}(1 + B\omega(z))^{\frac{\gamma-pB}{B}} & \text{for } B \neq 0, \\
  z^{p-q}e^{\gamma\omega(z)} & \text{for } B = 0.
\end{cases}
\]

Proof. By Lemma 2.2, the set of values \( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \) is obtained which comprises the closed disc with center at \( C(r) \) and the radius \( \rho(r) \), where

\[
C(r) = \frac{(p-q) - [B(\gamma - pB) + (p-q)B^2] r^2}{1 - B^2r^2} \quad \text{and} \quad \rho(r) = \frac{(\gamma - pB) r}{1 - B^2r^2}.
\]

(2.18)

Now by the definition of the class \( \mathcal{S}_q(A, B, p, \alpha) \) we have,

\[
\left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - C(r) \right| \leq \rho(r)
\]

(2.19)

This gives,

\[
\Re \left( \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right) \geq \frac{(p-q) - (\gamma - pB)r - [B(\gamma - pB) + (p-q)B^2] r^2}{1 - B^2r^2}
\]

(2.20)

Hence for \( r < R_{sc} \) the right hand side of the preceding inequality is positive, implying that

\[
R_{sc} \leq \frac{2(p-q)}{(\gamma - pB) + \sqrt{(\gamma - pB)^2 + 4(p-q)[(\gamma - pB) + (p-q)B^2]}}
\]

(2.21)

The radius is sharp, being attained by the function \( f_s^{(q)}(z) \) given by (2.17).

Remark 2.7. For parametric values of \( A, B, p \) and \( \alpha \) we get the well known results proved by Aouf, Nasr and also the results of Yasar Polatoglu.

References


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