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A Note on Radius of Starlikeness and Convexity of p - Valent Analytic Functions

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Abstract. Let $\mathcal{A}_p$ denote the class of analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, ...\})$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ and Ω denote the class of functions such that $\omega(0) = 0$ and $|\omega(z)| < 1$. Let $\mathcal{P}(A, B, p, \alpha)$ be the class of functions of the form

$$p(z) = p + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad p(z) = \frac{p + \gamma\omega(z)}{1 + B\omega(z)}, \ -1 \leq B < A \leq 1$$

where $\gamma = (p - \alpha)A + \alpha B$. In this paper, we define the class $\mathcal{S}_q(A, B, p, \alpha)$ of functions $f(z) \in \mathcal{A}_p$ such that

$$q + \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = p(z) \quad \text{for} \quad p(z) \in \mathcal{P}(A, B, p, \alpha)$$

and radius of starlikeness and convexity of functions in this class are studied.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$. Let Ω denote the class of bounded analytic functions $\omega(z)$ in $\mathcal{U}$ satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| \leq 1\,(z \in \mathcal{U})$.

For functions $g(z)$ and $G(z)$ analytic in $\mathcal{U}$, we say that $g(z)$ is subordinate to $G(z)$ if there exists a Schwarz function $\omega(z) \in \Omega$ such that $g(z) = G(\omega(z))$. If $G(z)$ is univalent in $\mathcal{U}$, then $g(z)$ is subordinate to $G(z)$ if and only if $g(0) = G(0)$ and $g(\mathcal{U}) \subset G(\mathcal{U})$.

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, $\mathcal{P}(A, B, p, \alpha)$ [1] denote the class of analytic functions defined in $\mathcal{U}$ such that

$$p(z) = p + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad p(z) = \frac{p + \gamma\omega(z)}{1 + B\omega(z)}, \ -1 \leq B < A \leq 1. \tag{1.2}$$

where $\gamma = (p - \alpha)A + \alpha B$. Further, $p(z) \in \mathcal{P}(A, B, p, \alpha)$ if and only if

$$p(z) = (p - \alpha)p_1(z) + \alpha, \quad p_1(z) \in \mathcal{P}(A, B) \tag{1.3}$$

where $\mathcal{P}(A, B)$ [3] is the Janowski class of functions $p_1(z)$ which are of the form

$$p_1(z) = 1 + \sum_{n=1}^{\infty} b_n z^n \tag{1.4}$$

and are analytic in $\mathcal{U}$, such that $p_1(z) \in \mathcal{P}(A, B)$ if and only if

$$p_1(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \ -1 \leq B < A \leq 1, \ \omega(z) \in \Omega, \ z \in \mathcal{U}. \tag{1.5}$$

We define the class $\mathcal{S}_q(A, B, p, \alpha)$ of functions $f(z) \in \mathcal{A}_p$ such that

$$q + \frac{z f^{(q+1)}(z)}{f^{(q)}(z)} = p(z) \quad \text{for} \quad p(z) \in \mathcal{P}(A, B, p, \alpha) \tag{1.6}$$

2. MAIN RESULTS

Lemma 2.1. **(Jack's Lemma)** [2]**:** *Let $\omega(z)$ be a regular function in the unit disc $\mathcal{U}$ with $\omega(0) = 0$, then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_1, we can write $z_1\omega'(z_1) = k\omega(z_1)$, where k is real and $k \geq 1$.*

Lemma 2.2. *The function*

$$\omega = \begin{cases} \dfrac{p+\gamma z}{1+Bz} & \text{for } B \neq 0, \\ \\ p+\gamma z & \text{for } B = 0 \end{cases} \tag{2.1}$$

maps $|z| = r$ onto a disc centered at $C(r)$ and having the radius $\rho(r)$ given by

$$C(r) = \begin{cases} \left(\dfrac{p-\gamma Br^2}{1-B^2r^2}, 0\right) & \text{for } B \neq 0, \\ \\ (p, 0) & \text{for } B = 0 \end{cases} \tag{2.2}$$

and

$$\rho(r) = \begin{cases} \dfrac{(\gamma - pB)r}{1-B^2r^2} & \text{for } B \neq 0, \\ \\ |\gamma| r & \text{for } B = 0. \end{cases} \tag{2.3}$$

Proof. Consider

(2.4)

$$\begin{cases} \omega = \dfrac{p+\gamma z}{1+Bz} \;\Leftrightarrow\; z = \dfrac{\omega - p}{\gamma - B\omega} \;\Leftrightarrow\; |z|^2 = r^2 = \dfrac{|\omega - p|^2}{|\gamma - B\omega|^2} & \text{for } B \neq 0 \\ \\ \Rightarrow\;\; u^2 + v^2 + \left(\dfrac{2\gamma Br^2 - 2p}{1-B^2r^2}\right) u + \dfrac{p^2 - \gamma^2 r^2}{1-B^2r^2} = 0, \quad \text{for } B \neq 0. \\ \\ \omega = 1 + \gamma z \;\Leftrightarrow\; z = \dfrac{\omega - p}{\gamma} \;\Leftrightarrow\; |z|^2 = r^2 = \dfrac{|\omega - p|^2}{|\gamma|^2} & \text{for } B = 0 \\ \\ \Rightarrow\;\; u^2 + v^2 - 2u + p^2 - \gamma^2 r^2 = 0, \quad \text{for } B = 0. \end{cases}$$

Lemma follows from (2.4). □

Lemma 2.3. *The function*

$$\omega = \begin{cases} \dfrac{(\gamma - pB)z}{1+Bz} & \text{for } B \neq 0, \\ \\ \gamma z & \text{for } B = 0 \end{cases} \tag{2.5}$$

maps $|z| = r$ *onto a disc centered at* $C(r)$ *and having the radius* $\rho(r)$ *given by*

$$C(r) = \begin{cases} \left(\dfrac{-B(\gamma - pB)r^2}{1 - B^2r^2}, 0\right) & \textit{for } B \neq 0, \\ (0,0) & \textit{for } B = 0 \end{cases} \tag{2.6}$$

and

$$\rho(r) = \begin{cases} \dfrac{(\gamma - pB)r^2}{1 - B^2r^2} & \textit{for } B \neq 0, \\ |\gamma| r & \textit{for } B = 0. \end{cases} \tag{2.7}$$

Proof. Consider

(2.8)
$$\begin{cases} \omega = \dfrac{(\gamma - pB)z}{1 + Bz} \;\Leftrightarrow\; z = \dfrac{\omega}{\gamma - pB - B\omega} \;\Leftrightarrow\; |z|^2 = r^2 = \dfrac{|\omega|^2}{|\gamma - pB - B\omega|^2} \quad \text{for } B \neq 0 \\ u^2 + v^2 + \left(\dfrac{2B(\gamma - pB)r^2}{1 - B^2r^2}\right) u + \dfrac{(\gamma - pB)^2 r^2}{1 - B^2r^2} = 0, \quad \text{for } B \neq 0. \\ \omega = \gamma z \;\Leftrightarrow\; z = \dfrac{\omega}{\gamma} \;\Rightarrow\; |z|^2 = r^2 = \dfrac{|\omega|^2}{|\gamma|^2} \;\Rightarrow\; u^2 + v^2 - \gamma^2 r^2 = 0 \text{ for } B = 0. \end{cases}$$

Lemma follows from (2.8). □

Theorem 2.4. *Let* $f(z) \in \mathcal{A}_p$ *be such that*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \prec \begin{cases} \dfrac{(\gamma - pB)z}{1 + Bz} = F_1(z) & \textit{for } B \neq 0, \\ \gamma z = F_2(z) & \textit{for } B = 0. \end{cases} \tag{2.9}$$

Then, $f(z) \in \mathcal{S}_q(A, B, p, \alpha)$ *and this result is sharp being obtained by the function* $\dfrac{p + \gamma z}{1 + Bz}$.

Proof. Define

$$\frac{f^{(q)}(z)}{z^{p-q}} = \begin{cases} (1 + B\omega(z))^{\frac{(\gamma - pB)}{B}} & \text{for } B \neq 0, \\ e^{\gamma \omega(z)} & \text{for } B = 0 \end{cases} \tag{2.10}$$

where $(1 + B\omega(z))^{\frac{(\gamma - pB)}{B}}$ and $e^{\gamma\omega(z)}$ have the value at 1 at the origin.
Then $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0) = 0$. On logarithmic differentiation we

get,

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \prec \begin{cases} \dfrac{(\gamma - pB)z\omega'(z)}{1 + B\omega(z)} & \text{for } B \neq 0, \\ \gamma z\omega'(z) & \text{for } B = 0. \end{cases} \tag{2.11}$$

Now it is easy to realize that subordination (2.9) is equivalent to $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. By Jack's Lemma it follows that, there exists a point $z_1 \in \mathcal{U}$ such that

(2.12)

$$\frac{z_1 f^{(q+1)}(z_1)}{f^{(q)}(z_1)} - p + q \prec \begin{cases} \dfrac{(\gamma - pB)k\omega(z_1)}{1 + B\omega(z_1)} = F_1(\omega(z_1)) \notin F_1(\mathcal{U}) & \text{for } B \neq 0, \\ \gamma k\omega(z_1) = F_2(\omega(z_1)) \notin F_2(\mathcal{U}) & \text{for } B = 0. \end{cases}$$

This contradicts our assumption given by (2.9) and the fact that $|\omega(z)| < 1$ for all $z \in \mathcal{U}$.

By using the condition (2.11), we get

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + q = \begin{cases} \dfrac{p + \gamma\omega(z)}{1 + B\omega(z)} & \text{for } B \neq 0, \\ p + \gamma\omega(z) & \text{for } B = 0. \end{cases} \tag{2.13}$$

Now by inequality (2.13) we obtain

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} + q \prec \begin{cases} \dfrac{p + \gamma z}{1 + Bz} & \text{for } B \neq 0, \\ p + \gamma z & \text{for } B = 0. \end{cases} \tag{2.14}$$

By inequality (2.14) it follows that $f(z) \in \mathcal{S}_q(A, B, p, \alpha)$. □

Corollary 2.5. *Let $f(z) \in \mathcal{S}_q(A, B, p, \alpha)$. Then, $f(z)$ can be written in the form*

$$f_*^{(q)}(z) = \begin{cases} z^{p-q}(1 + B\omega(z))^{\frac{\gamma - pB}{B}} & \text{for } B \neq 0, \\ z^{p-q}e^{\gamma\omega(z)} & \text{for } B = 0. \end{cases} \tag{2.15}$$

Theorem 2.6. *The radius of starlikeness and the radius of convexity of the $\mathcal{S}_q(A, B, p, \alpha)$ is given by*

$$R_{sc} = \frac{2(p - q)}{(\gamma - pB) + \sqrt{(\gamma - pB)^2 + 4(p - q)\left[(\gamma - pB) + (p - q)B^2\right]}}. \tag{2.16}$$

The radius is sharp, being attained by the function

$$f_*^{(q)}(z) = \begin{cases} z^{p-q}(1+B\omega(z))^{\frac{\gamma-pB}{B}} & \text{for } B \neq 0, \\ z^{p-q}e^{\gamma\omega(z)} & \text{for } B = 0. \end{cases} \tag{2.17}$$

Proof. By Lemma 2.2, the set of values $\dfrac{zf^{(q+1)}(z)}{f^{(q)}(z)}$ is obtained which comprises the closed disc with center at $C(r)$ and the radius $\rho(r)$, where

$$C(r) = \frac{(p-q) - \left[B(\gamma - pB) + (p-q)B^2\right] r^2}{1 - B^2r^2} \quad \text{and} \quad \rho(r) = \frac{(\gamma - pB)r}{1 - B^2r^2}. \tag{2.18}$$

Now by the definition of the class $\mathcal{S}_q(A, B, p, \alpha)$ we have,

$$\left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - C(r) \right| \leq \rho(r). \tag{2.19}$$

This gives,

$$\Re\left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)}\right) \geq \frac{(p-q) - (\gamma - pB)r - \left[B(\gamma - pB) + (p-q)B^2\right] r^2}{1 - B^2r^2}. \tag{2.20}$$

Hence for $r < R_sc$ the right hand side of the preceding inequality is positive, implying that

$$R_{sc} \leq \frac{2(p-q)}{(\gamma - pB) + \sqrt{(\gamma - pB)^2 + 4(p-q)\left[(\gamma - pB) + (p-q)B^2\right]}}. \tag{2.21}$$

The radius is sharp, being attained by the function $f_*^{(q)}(z)$ given by (2.17). □

Remark 2.7. *For parametric values of* A, B, p *and* α *we get the well known results proved by Aouf , Nasr and also the results of Yasar Polatoglu.*

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