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Neighborhood Connected Equitable Domination in Graphs

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Abstract

Let $G = (V, E)$ be a connected graph, An equitable dominating S of a graph G is called the neighborhood connected equitable dominating set (nced-set) if the induced subgraph $\langle N_e(S) \rangle$ is connected The minimum cardinality of a nced-set of G is called the neighborhood connected equitable domination number of G and is denoted by $\gamma_{nce}(G)$. In this paper we initiate a study of this parameter. For any graph G.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected with neither loops nor multiple edges the order and size of G are denoted by p and q respectively for graph theoretic terminology we refer to Chartrand and Lesnaik [2] A subset S of V is called a dominating set if $N[S] = V$ the minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$, ($\Gamma(G)$). An excellent treatment of the fundamentals of domination is given in the book by Haynes etal [4] A survey of several advanced topics in domination is given in the book edited by Haynes et al. [3]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [3]. Sampathkumar and Walikar [5] introduced the concept of connected domination in graphs. Let $G = (V, E)$ be a graph and let $v \in V$ the open neighborhood and the closed neighborhood of v are denoted by $N(v)$ and $N[V] = N(v) \cup v$ respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $Pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.

A dominating set S of G is called a connected dominating set if the induced subgraph $< \langle S \rangle$ is connected the minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S) \rangle$ is connected. The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$. A ncd-set S is said to be minimal if no proper subset of S is a ncd-set. A coloring of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum integer K for which a graph G is $k-$colorable is called the chromatic number of G and is denoted by $\chi(G)$.

A subset S of V is called an equitable dominating set if for every $v \in V - S$ there exist a vertex $u \in S$ such that $uv \in E(G)$ and $|d(u) - d(v)| \leq 1$. The minimum cardinality of such a equitable dominating set is denoted by γ_e and is called the equitable domination number of G. A vertex u V is said to be degree equitable with a vertex $v \in V$ if $|d(u) - d(v)| \leq 1$. If S is an equitable dominating set then any super set of S is an equitable dominating set. An equitable set S is said to be a minimal equitable dominating set if no proper subset of S is an equitable dominating set. The minimal upper equitable dominating number is Γ_e the upper equitable dominating set of G. If $u \in V$ such that $|d(u) - d(v)| \geq 2$ for every $v \in N(u)$ then u is in every equitable dominating set such points are called equitable isolates. I_e denotes the set of all equitable isolates. An equitable dominating S of connected graph G is called an equitable connected dominating set (ecd-set) if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a ecd-set of G is called the equitable connected domination number of G and is denoted by $\gamma_{ec}(G)$. Let $G = (V, E)$ be a graph and let $u \in V$ the equitable neighborhood of u denoted by $N_e(u)$ is defined as $N_e(u) = \{v \in V : |v \in N(u), |d(u) - d(v)| \leq 1\}$. The maximum and minimum equitable degree of a point in G are denoted by $\Delta_e(G)$ and $\delta_e(G)$ that is $\Delta_e(G) = max_{u \in V(G)} |N_e(u)|$ and $\delta_e(G) = min_{u \in V(G)} |N_e(u)|$.

The open equitable neighbourhood and closed equitable neighbourhood of v are denoted by $N_e(v)$ and $N_e[v] = N_e(v) \cup \{v\}$ respectively. If $S \subseteq V$ then $N_e(S) = \cup_{v \in S} N_e(v)$ and $N[S] = N_e(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private equitable neighbor set of u with respect to S is defined by $pne[u, S] = N_e[u] - N_e[S - \{u\}]$.

2 Main Results

Definition. An equitable dominating set S of a graph G is called the neighborhood connected equitable dominating set (nced-set) if the induced subgraph $\langle N_e(S) \rangle$ is connected the minimum cardinality of a nced-set of G is called the neighborhood connected equitable domination number of G and is denoted by $\gamma_{nce}(G)$.

Examples: γ_{nce} value for well known graphs

1)$\gamma_{nce}(K_p) = 1$

2)$\gamma_{nce}(P_p) = \lceil \frac{p}{2} \rceil$

3)$\gamma_{nce}(C_p)=\begin{cases} \lfloor \frac{p}{2} \rfloor, & \text{if } p \equiv 3(mod4); \\ \lceil \frac{p}{2} \rceil, & \text{otherwise.} \end{cases}$

4)$\gamma_{nce}(W_p)=\begin{cases} \lceil \frac{p-1}{2} + 1 \rceil, & \text{if } p \equiv 3(mod4); \\ 1, & \text{otherwise.} \end{cases}$

5)$\gamma_{nce}(K_{r,t})=\begin{cases} 2, & \text{if } |r-t| \leq 1; \\ r+t, & \text{if } |r-t| \geq 2 \text{ and } r,t \geq 2. \end{cases}$

In the following proposition we determine the relation between the $\gamma_{nce}(G)$ and the others invariant domination parameters.

Theorem 2.1 *For any graph G 1.* $\gamma(G) \leq \gamma_{nc}(G) \leq \gamma_{nce}(G)$.

2. $\gamma(G) \leq \gamma_{nce}(G) \leq 2\gamma(G)$.

3. $\gamma_{nce}(G) \leq \gamma_{ec}(G)$.

4. $\gamma(G) \leq \gamma_e(G)$.

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Theorem 2.2 *For any path* P_p, $\gamma_{nce}(P_p) = \lceil \frac{p}{2} \rceil$.

Proof. Let $P_p = \{v_1, v_{2,\ldots,v_p}\}$. If $p \;\; not \equiv 1(mod4)$. Then $S = \{v_{j:j=2k,2k+1 \; and \; k \; is \;\; odd}\}$ is a nced-set of P_p and if $p \equiv 1(mod4)$, then $S_i = S \cup \{v_{p-1}\}$ is a nced-set of

P_p. Hence, $\gamma_{nce}(P_p) \leq \lceil \frac{p}{2} \rceil$. Since $\gamma_{nc}(P_p) = \lceil \frac{p}{2} \rceil$ and $\gamma_{nc}(G) \leq \gamma_{nce}(G)$. We have $\lceil \frac{p}{2} \rceil \leq \gamma_{nce}(G)$. Thus $\gamma_{nce}(P_p) = \lceil \frac{p}{2} \rceil$.

Corollary 2.2.1. For any non-trivial path P_p

a)$\gamma_{nce}(P_p) = \gamma(P_p)$ if and only if p=2 or 4

b)$\gamma_{nce}(P_p) = \gamma_e(P_p)$ if and only if p=2 or 4

Proof. Since $\gamma(P_p) = \gamma_e(P_p) = \lceil \frac{p}{2} \rceil$ the corollary follows.

Theorem 2.3 $\gamma_{nce}(C_p)=\begin{cases} \lfloor \frac{p}{2} \rfloor, & \textit{if } p \equiv 3(mod4); \\ \lceil \frac{p}{2} \rceil, & \textit{otherwise.} \end{cases}$

Proof. Let $V(C_p) = \{v_1, v_2, ..., v_p\}$ and $p = 4k + r$ where, $0 \leq r \leq 3$.

Let $S_1=\begin{cases} S, & \text{if } p \equiv 0(mod4); \\ S \cup \{v_p\}, & p \equiv 1 \text{ } or \text{ } 2(mod4); \\ S \cup \{v_{p-1}\}, & p \equiv 3(mod4). \end{cases}$

Clearly S_1 is nced-set of C_p and hence

$$\gamma_{nce}(C_p) \leq \begin{cases} \lfloor \frac{p}{2} \rfloor, & \text{if } p \equiv 3(mod4); \\ \lceil \frac{p}{2} \rceil, & \text{otherwise.} \end{cases}$$

Now Let S be any γ_{nce} - set of C_p then $< S >$ contains at most one isolated vertex.

And $\langle N_e(S) \rangle=\begin{cases} C_p, & p \equiv 0(mod4); \\ P_{p-1}, & \text{otherwise.} \end{cases}$

Hence

$$|S| \geq \gamma_{nce}(C_p)=\begin{cases} \lfloor \frac{p}{2} \rfloor, & \text{if } p \equiv 3(mod4); \\ \lceil \frac{p}{2} \rceil, & \text{otherwise.} \end{cases}$$

And the result follows.

Corollary 2.3.1.

1) $\gamma_{nce}(C_p) = \gamma(C_p)$ if and only if $p = 3, 4$ or 7

2) $\gamma_{nce}(C_p) = \gamma_e(C_p)$ if and only if $p = 3, 4$ or 5, 7, 8

3) $\gamma_{nce}(C_p) = \gamma_{nc}(C_p)$ if and only if $p \equiv 0, 1, 3, (mod4)$.

Proof. Since $\gamma(C_p) = \lceil \frac{p}{3} \rceil$,$\gamma_{nc}(C_p) = p - 2$

$$\gamma_e(C_p)=\begin{cases} \lceil \frac{p}{3} + 1 \rceil, & \text{if } p \equiv 2(mod3); \\ \lfloor \frac{p}{3} \rfloor, & \text{otherwise.} \end{cases}$$

$$\gamma_{nc}(C_p)=\begin{cases} \lfloor \frac{p}{2} \rfloor, & \text{if } p \equiv 3(mod4); \\ \lceil \frac{p}{3} \rceil, & \text{otherwise.} \end{cases}$$

The result follows

Lemma A. A super set of a nced-set is a nced-set.

Proof. Let S be a nced-set of a graph G and let $S_1 = S \cup \{v\}$ where, $v \in V - S$. Clearly $v \in N_e(S)$ and S_1 is a dominating set of G. Now let $x, y \in N_e(S_1)$. If $x, y \in N_e(S)$, then any $x - y$ path in $N_e(S)$ is a $x - y$ path in $N_e(S_1)$. If $x \in N_e(S)$ and $y \notin Ne(S)$, then $y \in N_e(v)$ and any $x - v$ path in $N_e(S)$ followed by the edge vy is $x - y$ path in $N_e(S_1)$. Also if $x, y \notin N_e(S)$ then (x, v, y) is a $x - y$ path in $N_e(S_1)$. Thus $\langle N_e(S_1) \rangle$ is connected so that S_1 is a nced-set of G.

Theorem 2.4 *A nced-set S of a graph G is minimal nced-set if and only if for every $u \in S$ one of the following holds*

1)$pne[u, S] \neq \phi$

2)There exist two vertices $x, y \in N_e(S)$ such that every $x-y$ path in $\langle N_e(S) \rangle$ contains at least one vertex of $N_e(S) - N_e(S - \{u\})$.

Proof. Let S be a minimal nced-set of G, Let $u \in S$ and let $S_1 = S \cup \{u\}$ is disconnected. If S_1 is not a dominating set of G or $\langle N_e(S_1) \rangle$ is disconnected. If S_1 is not a dominating set of G, then $pne[u, S] \neq \phi$. If $\langle N_e(S_1) \rangle$ is disconnected. Then there exist two vertices $x, y \in N_e(S_1)$ such that there is no $x - y$ path in $\langle N_e(S_1) \rangle$ since $\langle N_e(S) \rangle$ connected, it follows that every $x - y$ path in $\langle N_e(S) \rangle$ contains at least one vertex of $N_e(S) - N_e(S - \{u\})$ conversely, if S is a nced-set of G satisfying the conditions of the theorem.

Theorem 2.5 *Let G be a graph with $\Delta_e = p - 1$ then $\gamma_{nce}(G) = 1$ or 2 further $\gamma_{nce}(G) = 2$ if and only if G has exactly one vertex v with $deg_e(v) = p - 1$ and v is a cut vertex of G.*

Proof. Let $v \in V(G)$ and $deg_e(v) = p - 1$. Then $\{u, v\}$ where $u \in V - \{v\}$ is a nced-set of G so that $\gamma_{nce}(G) \leq 2$ now suppose $\gamma_{nce}(G) = 2$, then $\langle N_e(v) \rangle = G - v$ is disconnected and hence v is a cutvertex of G. Hence it follows that v is only vertex of G with $deg_e(v) = p - 1$. The converse is obvious.

Theorem 2.6 *Let G be a graph without any equitable isolated point with $\Delta_e \leq p - 1$. Then $\gamma_{nce}(G) \leq p - \Delta_e$.*

Proof. Let $v \in V(G)$ and $deg_e(v) = \Delta_e$. Since G is connected and $\Delta_e \leq p - 1$ there exist two adjacent vertices u and w such that$u \in N_e(v)$ and $w \notin N_e(v)$. Now let$S = (N_e(v) - \{u\}) \cup \{w\}$. Clearly $V - S$ is a nced-set of G and hence $\gamma_{nce}(G) \leq p - \Delta_e$.

Theorem 2.7 *Let $G = (V, E)$ be a graph such that $V = \{v_1, ..., v_p\}$ and $deg_e(v_i) \geq 1$ and G have k pendant vertices. Then $\gamma_{nce}(G) \leq p - k$.*

Proof. Let X be the set of all pendant vertices of a graph G. Let $|X| = k$. Then $(V - X)$ is nced-set of G.
Hence $\gamma_{nce}(G) \leq p - k$

A graph with $\gamma_{nce}(G) \leq p - k$

Theorem 2.8 *Let G be a graph with $\Delta_e = p - 1$ and let $v \in V(G)$ with $deg_e(v) = \Delta_e$ then $\gamma_{nce}(G) \leq 1 + |V(H)|$, where H is a component of $G - v$ with $|V(H)|$ is minimum.*

Proof. Let $v \in V(G)$ with $deg_e(v) = p - 1$. If $G - v$ is connected then $\{v\}$ is a nced-set of G and hence $\gamma_{nce}(G) = 1$ suppose $G - v$ is disconnected then $S = \{v\}$ is not a nced-set of G. Let H be a component of $G - v$ with minimum vertices. Hence $S \cup (V(H)$ is a nced-set of G. Thus $\gamma_{nce}(G) \leq 1 + |V(H)|$.
Remark: The bound given in above is sharp the graph $G = K_{1,2}$, $\gamma_{nce} = 2 = 1 + |V(H)|$.
Corollary 2.8.1. Let G be a graph with $\Lambda_e = p - 1$. Then $\gamma_{nce}(G) = 2$ if and only if there exists a support vertex v such that $deg_e(v) = p - 1$.

Theorem 2.9 *If T is non-trivial tree, then $\gamma_{nce}(T) \leq p - 1$ such that T is not a star.*

Proof. Since T is non-trivial tree this implies that $V - v$ is a nced-set of T. Thus it holds

Theorem 2.10 *For any graph$\gamma_{nce}(G) \leq \lceil \frac{p}{2} \rceil$ if G has no equitable isolated vertex.*

Proof. If T is any spanning tree then $\gamma_{nce}(G) \leq \gamma_{nce}(T)$. It is enough to prove the result for trees which we prove by induction on p obviously the result is true when $p = 2$ or 3 we now assume that the result is true for all trees of order less than p and let T be a tree of order $p \geq 4$. If p is odd let $T_1 = T - \{v\}$ where, v is a pendant vertex of T then $\gamma_{nce}(T_1) \leq \frac{p-1}{2}$ so that$\gamma_{nce}(T) \leq \gamma_{nce}(T_1) + 1 \leq \lceil \frac{p}{2} \rceil$.

Theorem 2.11 *For any graph G with $\Delta_e \geq 1$, $\gamma_{nce}(G) \leq 2q - p + 1$.*

Proof. clearly from the definition of the neighbourhood connected equitable dominating set we have $\gamma_{nce}(G) \leq p - 1$, then $\gamma_{nce}(G) \leq p - 1 = 2(p - 1) - (p - 1) \leq 2q - p + 1$.

Theorem 2.12 *For any non-trivial graph G, $\gamma_{nce}(G) + \chi(G) \leq 2p - 1$ and equality holds if and only if G is isomorphic to K_2.*

Proof. If $\gamma_{nce}(G) + \chi(G) = 2p$, then $\gamma_{nce}(G) = p$ and $\chi(G) = p$ then G is a complete graph with $\gamma_{nce}(G) = p$ which gives G is trivial and hence $\gamma_{nce}(G) + \chi(G) \leq 2p - 1$.
Let G be a graph with $\gamma_{nce}(G)+\chi(G) = 2p-1$. Then either (i) $\gamma_{nce}(G) = p-1$, $\chi(G) = p$ or (ii) $\gamma_{nce}(G) = p$, $\chi(G) = p-1$. If (i) is holds then G is a complete graph with $\gamma_{nce}(G) = p - 1$ which gives $p = 2$. Hence G is isomorphic to K_2.

If (ii) holds then G is a isomorphic to $K_p - X$ where X is non empty subset of set of edges incident with a vertex v of K_p with $|X| \leq p - 2$ which implies $\gamma_{nce}(G) = 1$ or 2. Then $p = 2$ and hence G is disconnected which is a contradiction. The converse is obvious.

Theorem 2.13 *Let G be a graph. Then $\gamma_{nce}(G) + \chi(G) = 2p - 2$ if and only if G is isomorphic to K_3 or P_3 or the graph obtained from $k \cup H$ where $K = K_{n-2}$ and H is either K_2 or $\overline{K_2}$ with $V(H) = \{u, v\}$ by adding p_1 edges between u and K and adding p_2 edges between v and K, $2 \leq p_i \leq p - 5$, $i = 1$ or 2, such that $[N_e(u) \cap N_e(v)] - \{u, v\} = \phi$ and $p_1 + p_2 < p - 2$.*

Proof. Let $\gamma_{nce}(G) + \chi(G) = 2p - 2$. Then one of the following is true (i) $\gamma_{nce}(G) = p-2$,$\chi(G) = p$, (ii) $\gamma_{nce}(G) = p-1$, $\chi(G) = p-1$, (iii) $\gamma_{nce}(G) = p$, $\chi(G) = p - 2$.

If G is complete graph then (i) holds such that $\gamma_{nce}(G) = p-2$ this implies $p = 3$. Hence G is isomorphic to K_3.
If G is isomorphic $K_p - X$ where X is a non empty subset of set of edges incident with a vertex of K_p with $|X| \leq p - 2$ which implies $\gamma_{nce}(G) = 1$ or 2 then $p = 2$ or 3 and hence G is isomorphic to P_3.
Suppose (iii) holds. Because $\chi(G) = p - 2$ either G has a complete subgraph of order $p - 2$ or $p > 4$ and G is the join of K_{p-5} with C_5 (in case $p = 5$ by the join of K_{p-5} and C_5 we mean C_5'). If G is the join of K_{p-5} with C_5 then $\gamma_{nce}(G) + \chi(G) = 6$ if $p = 5$ or $p - 1$ if $p > 5$. In either case, $\gamma_{nce}(G) + \chi(G) \neq 2p - 2$. Thus G has a complete subgraph G_1 of order $p - 2$. Let $Y = V(G) - V(G1) = \{u, v\}$. Then $\langle Y\rangle = K_2$ or $\overline{K_2}$.
. **Case 1:** $\langle Y\rangle = \overline{K_2}$
Since G is a connected graph each u and v are equitable adjacent to at least one vertex of G_1. If either u or v is a pendant vertex then $\gamma_{nce}(G) < p$. Hence each u and v are adjacent to at least two vertices in G_1. If u and v have a common neighbor w in G_1, then $\gamma_{nce}(G) = 1$ which gives a contradiction. Hence $N_e(u) \cap N_e(v) = \phi$. If $N_e(u) \cap N_e(v) = V(G1)$ then $\gamma_{nce}(G) = 2$ which is a contradiction. Then the graph is isomorphic to the graph given in the theorem.
Case 2: $\langle Y\rangle = K_2$
Since G is connected and $\gamma_{nce}(G) = p$ we have each u and v are equitable

adjacent to at least one vertex of G_1. If u and v have a common neighbor w in G_1, then $\gamma_{nce}(G) = 1$ or 2 which gives a contradiction. Hence $N_e(u) \cap N_e(v) = \phi$ suppose $N_e(u) \cap N_e(v) = x$ then $\{u, x\}$ is a γ_{nce}-set G which is a contradiction. Hence each u and v are adjacent to more than one vertex in G_1. If $[N_e(u) \cup N_e(v)] - \{u, v\} = V(G_1)$ then $\gamma_{nce}(G) = 2$ which is a contradiction then the graph is isomorphic to the graph in the theorem. The Converse is obvious.

Nordhaus Gaddum type result:

Theorem 2.14 *For any graph G, $\gamma_{nce}(G) + \gamma_{nce}(\overline{G}) \leq (p-1)(p-2)$.*

Proof. Since from Proposition 2.11 we have

$$\gamma_{nce}(G) \leq 2q - p + 1$$
$$\gamma_{nce}(\overline{G}) \leq 2\overline{q} - p + 1$$

Now

$$\gamma_{nce}(G) + \gamma_{nce}(\overline{G}) \leq (2\overline{q} - p + 1) + (2q - p + 1)$$

Hence $\gamma_{nce}(G) + \gamma_{nce}(\overline{G}) \leq (p-1)(p-2)$.

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