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A Note on Multiplier Transformation

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Abstract. In the present paper,we define the class $\mathcal{T}_\lambda^m(\alpha,\beta,l)$ using the multiplier transformation. For functions belonging to this class we discuss coefficient estimates, inclusion relations, extreme points and some more properties.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$.

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathcal{U}$, consisting of analytic functions whose non-zero coefficients from the second onwards are negative. That is, an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.2}$$

which are analytic in the open unit disc $\mathcal{U}$.

For $f \in \mathcal{A}$, multiplier transformation $I(m,\lambda,l)f(z)$ [3] defined by

$$I(m,\lambda,l)f(z) = z - \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k z^k, m \in N_0, \lambda, l \geq 0, j \in \mathbb{N}$$

Now using multiplier transformation, we define the subclass of $\mathcal{T}$.

Let $\mathcal{T}_\lambda^m(\alpha,\beta,l)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the conditions

$$\Re\left\{\frac{z((I(m,\lambda,l)f)'}{\beta z(I(m,\lambda,l)f)' + (1-\beta)I(m,\lambda,l)f}\right\} > \alpha, \tag{1.3}$$

for some α, β $(0 \leq \alpha, \beta < 1)$, $\lambda, l \geq 0$ and $m \in \mathbf{N}_0$.

2. Main Results

Theorem 2.1. *A function f defined by (1.2) is in the class $\mathcal{T}_\lambda^m(\alpha,\beta,l)$ if and only if*

$$\sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[k - \alpha\beta k + \alpha\beta - \alpha] < 1-\alpha, \tag{2.1}$$

where α, β $(0 \leq \alpha, \beta < 1)$, $\lambda, l > 0$ and $m \in \mathbf{N}_0$.

Proof. Suppose $f \in \mathcal{T}_\lambda^m(\alpha,\beta,l)$. Then

$$\Re\left\{\frac{z(I(m,\lambda,l)f)'}{\beta z(I(m,\lambda,l)f)' + (1-\beta)I(m,\lambda,l)f}\right\} > \alpha,$$

$$\Re\left\{\frac{z - \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m k a_k z^k}{\beta[z - \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m k a_k z^k] + (1-\beta)[z - \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k z^k]}\right\} > \alpha,$$

$$\Re\left\{\frac{z - \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m k a_k z^k}{z - \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k z^k[\beta(k-1)+1]}\right\} > \alpha.$$

Letting $z \to 1$, we get,

$$1-\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m ka_k > \alpha\left\{1-\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[\beta(k-1)+1]\right\}.$$

Equivalenty we have,

$$\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m ka_k-\alpha\left\{\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[\beta(k-1)+1]\right\} < (1-\alpha)$$

which implies

$$\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[k-\alpha\beta k+\alpha\beta-\alpha] < (1-\alpha).$$

Conversely, assume that (2.1) is true. We have to show that (1.3) is satisfied or equivalently

$$\left|\left\{\frac{z(I(m,\lambda,l)f)'}{\beta z(I(m,\lambda,l)f)'+(1-\beta)I(m,\lambda,l)f}\right\}-1\right| < 1-\alpha.$$

Where

$$\left|\left\{\frac{z-\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m ka_kz^k}{z-\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_kz^k[\beta(k-1)+1]}\right\}-1\right|$$

$$=\left|\frac{\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k(k-1)(\beta-1)z^k}{z-\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[\beta(k-1)+1]z^k}\right|$$

$$\leq\frac{\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k(k-1)(\beta-1)|z^k|}{|z|-\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[\beta(k-1)+1]|z^k|}$$

$$\leq\frac{\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k(k-1)(\beta-1)}{1-\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[\beta(k-1)+1]}.$$

is bounded above by $1-\alpha$ if

$$\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k(k-1)(\beta-1) \le (1-\alpha)(1-\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[\beta(k-1)+1])$$

or

$$\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[k-\alpha\beta k+\alpha\beta-\alpha] < 1-\alpha,$$

which is true by hypothesis. This completes the proof. □

For $l = o$, we get Theorem 2.1 of [5] as Corollary.

Corollary 2.2. *A function f defined by (1.2) is in the class $T_\lambda^m(\alpha,\beta,0)$ if and only if*

$$\sum_{k=j+1}^{\infty}[1+(k-1)\lambda]^m a_k[k-\alpha\beta k+\alpha\beta-\alpha] < 1-\alpha.$$

where $\alpha,\beta(0\le\alpha,\beta<1), \lambda>0$ *and* $m\in N_0$.

For $\lambda=1, m=0$ and $m=1,\lambda=1$ and $l=0$ respectively in theorem 2.1 we have the following result of Mostafa [4].

Corollary 2.3. i. *A function $f(z)$ defined by (1.2) is in the class $\mathcal{T}_1^0(\alpha,\beta,l)$ if and only if*

$$\sum_{k=j+1}^{\infty}(k-\alpha\beta k+\alpha\beta-\alpha)a_k \le 1-\alpha.$$

ii. *A function $f(z)$ defined by (1.2) is in the class $\mathcal{T}_1^1(\alpha,\beta,0)$ if and only if*

$$\sum_{k=j+1}^{\infty}k(k-\alpha\beta k+\alpha\beta-\alpha)a_k \le 1-\alpha.$$

Corollary 2.4. *If $f\in\mathcal{T}_\lambda^m(\alpha,\beta,l)$, then*

$$|a_k| \le \frac{1-\alpha}{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m[k-\alpha\beta k+\alpha\beta-\alpha]}.$$

Theorem 2.5. *Let* $0\le\alpha<1$, $0\le\beta_1\le\beta_2<1$, $n\in\mathbb{N}_0$, *then* $\mathcal{T}_\lambda^m(\alpha,\beta_2,l)\subset\mathcal{T}_\lambda^m(\alpha,\beta_1,l)$.

Proof. For $f \in \mathcal{T}_\lambda^m(\alpha, \beta_2, l)$. We have,

$$\sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[k - \alpha\beta_2 k + \alpha\beta_2 - \alpha]$$

$$\leq \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m a_k[k - \alpha\beta_1 k + \alpha\beta_1 - \alpha] < 1 - \alpha.$$

Hence $f \in \mathcal{T}_\lambda^m(\alpha, \beta_1, l)$. □

Theorem 2.6. *Let* $f \in \mathcal{T}_\lambda^m(\alpha, \beta, l)$. *Define* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{1-\alpha}{\sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k - \alpha\beta k + \alpha\beta - \alpha]} z^k, \quad k = 2, 3, \cdots,$$

for some α, $\beta\,(0 \leq \beta < 1), n \in \mathbb{N}_0$, $\lambda > 0$ *and* $z \in \mathcal{U}$. $f \in \mathcal{T}_\lambda^m(\alpha, \beta, l)$ *if and only if* f *can be expressed as* $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ *where* $\mu_k \geq 0$ *and* $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. If $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ with $\sum_{k=1}^{\infty} \mu_k = 1, \quad \mu_k \geq 0$, then

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} \frac{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k - \alpha\beta k + \alpha\beta - \alpha]\mu_k}{\frac{\lambda(k-1)+l+1}{l+1}^m [k - \alpha\beta k + \alpha\beta - \alpha]}(1-\alpha) \\
&\sum_{k=j+1}^{\infty} \mu_k(1-\alpha) = (1-\mu_1)(1-\alpha) \\
&\leq (1-\alpha).
\end{aligned}$$

Hence $f \in \mathcal{T}_\lambda^m(\alpha, \beta, l)$.

Conversely, let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{T}_\lambda^m(\alpha, \beta, l)$, define

$$\mu_k = \frac{\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k - \alpha\beta k + \alpha\beta - \alpha]\,|a_k|}{(1-\alpha)}, \quad k = 2, 3, \cdots,$$

and define $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$. From Theorem 2.1, $\sum_{k=2}^{\infty} \mu_k \leq 1$ and hence $\mu_1 \geq 0$.
Since $\mu_k f_k(z) = \mu_k f(z) + a_k z^k$,
$\sum_{k=1}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} a_k z^k = f(z)$. □

Theorem 2.7. *The class $\mathcal{T}_\lambda^m(\alpha, \beta, l)$ is closed under convex linear combination.*

Proof. Let $f,\ g \in \mathcal{T}_\lambda^m(\alpha, \beta, l)$ and let

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k.$$

For η such that $0 \leq \eta \leq 1$, it suffices to show that the function defined by $h(z) = (1-\eta) f(z) + \eta g(z), \quad z \in \mathcal{U}$ belongs to $\mathcal{T}_\lambda^m(\alpha, \beta, l)$. Now

$$h(z) = z - \sum_{k=2}^{\infty} [(1-\eta) a_k + \eta b_k] z^k.$$

Applying Theorem 2.1, to $f,\ g \in \mathcal{T}_\lambda^m(\alpha, \beta, l)$, we have

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k - \alpha\beta k + \alpha\beta - \alpha]\,[(1-\eta)a_k + \eta b_k] \\
&\quad = (1-\eta) \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k - \alpha\beta k + \alpha\beta - \alpha] a_k \\
&+ \eta \sum_{k=j+1}^{\infty} \left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k - \alpha\beta k + \alpha\beta - \alpha] b_k \\
&\quad \leq (1-\eta)(1-\alpha) + \eta(1-\alpha) = (1-\alpha).
\end{aligned}$$

This implies that $h \in \mathcal{T}_\lambda^m(\alpha, \beta, l)$. □

Corollary 2.8. *If $f_1(z),\ f_2(z)$ are in $\mathcal{T}_\lambda^m(\alpha, \beta, l)$ then the function defined by $g(z) = \frac{1}{2}[f_1(z) + f_2(z)]$ is also in $\mathcal{T}_\lambda^m(\alpha, \beta, l)$.*

Theorem 2.9. *Let for $i = 1, 2, \cdots, k, \quad f_i(z) = z - \sum_{m=k}^{\infty} a_{k,i} z^k \in \mathcal{T}_\lambda^m(\alpha, \beta, l)$ and $0 < \beta_i < 1$ such that $\sum_{i=1}^{k} \beta_i = 1$, then the function $F(z)$ defined by $F(z) = \sum_{i=1}^{k} \beta_i f_i(z)$ is also in $\mathcal{T}_\lambda^m(\alpha, \beta, l)$.*

Proof. For each $i \in \{1,2,3,\cdots,k\}$ we obtain

$$\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k-\alpha\beta k+\alpha\beta-\alpha]|a_k| < (1-\alpha).$$

Since

$$\begin{aligned} F(z) &= \sum_{i=1}^{k}\beta_i(z-\sum_{k=j+1}^{\infty}a_{k,i}z^k) \\ &= z-\sum_{k=j+1}^{\infty}(\sum_{i=1}^{k}\beta_i a_{k,i})z^k. \end{aligned}$$

$$\begin{aligned} &\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k-\alpha\beta k+\alpha\beta-\alpha]\left[\sum_{i=1}^{k}\beta_i a_{k,i}\right] \\ &\quad = \sum_{i=1}^{k}\beta_i\left[\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^m [k-\alpha\beta k+\alpha\beta-\alpha]\right] \\ &\quad < \sum_{i=1}^{k}\beta_i(1-\alpha) < (1-\alpha). \end{aligned}$$

Therefore $F(z) \in \mathcal{T}_\lambda^m(\alpha,\beta,l)$. □

Theorem 2.10. *Let* $f \in \mathcal{T}_\lambda^m(\alpha,\beta,l)$*. The Komato operator of* f *is defined by*

$$k(z) = \int_0^1 \frac{(c+1)^\gamma}{\Gamma(\gamma)} t^c \left(\log\frac{1}{t}\right)^{\gamma-1}\frac{f(tz)}{t}dt,$$

$c > -1, \quad \gamma \geq 0$ *then* $k(z) \in \mathcal{T}_\lambda^m(\alpha,\beta,l)$.

Proof. We have

$$\begin{aligned} &\int_0^1 t^c\left(\log\frac{1}{t}\right)^{\gamma-1}dt = \frac{\Gamma(\gamma)}{(c+1)^\gamma} \\ &\int_0^1 t^{k+c-1}\left(\log\frac{1}{t}\right)^{\gamma-1}dt = \frac{\Gamma(\gamma)}{(c+1)^\gamma}, \quad k=2,3,\cdots, \end{aligned}$$

$$\begin{aligned} k(z) &= \frac{(c+1)^\gamma}{\Gamma(\gamma)}\left[\int_0^1 t^c\left(\log\frac{1}{t}\right)^{\gamma-1}zdt - \sum_{k=j+1}^{\infty} z^k\int_0^1 a_k t^{k+c-1}\left(\log\frac{1}{t}\right)^{\gamma-1}dt\right] \\ &= z-\sum_{k=j+1}^{\infty}\left(\frac{c+1}{c+k}\right)^\gamma a_k z^k. \end{aligned}$$

Since $f \in \mathcal{T}_{\lambda}^{m}(\alpha,\beta,l)$ and since $\left(\dfrac{c+1}{c+k}\right)^{\gamma} < 1$, we have

$$\sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^{m}[k-\alpha\beta k+\alpha\beta-\alpha]\left(\frac{c+1}{c+k}\right)^{\gamma}a_k < (1-\alpha).$$

□

Theorem 2.11. *Let* $f \in \mathcal{T}_{\lambda}^{m}(\alpha,\beta,l)$, *then for every* $0 \le \delta < 1$ *the function*

$$\mathcal{H}_{\delta}(z) = (1-\delta)f(z) + \delta\int_0^z \frac{f(t)}{t}dt.$$

Proof. We have $\mathcal{H}_{\delta}(z) = z - \sum_{k=j+1}^{\infty}\left(1+\frac{\delta}{k}-\delta\right)a_k z^k.$

Since $\left(1+\dfrac{\delta}{k}-\delta\right) < 1, \quad k \ge 2,$ so by Theorem 2.1,

$$\begin{aligned}\sum_{k=j+1}^{\infty}\left(1+\frac{\delta}{k}-\delta\right)\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^{m}&[k-\alpha\beta k+\alpha\beta-\alpha]a_k\\ &< \sum_{k=j+1}^{\infty}\left(\frac{\lambda(k-1)+l+1}{l+1}\right)^{m}[k-\alpha\beta k+\alpha\beta-\alpha]a_k\\ &< (1-\alpha).\end{aligned}$$

Therefore $\mathcal{H}_{\delta}(z) \in \mathcal{T}_{\lambda}^{m}(\alpha,\beta,l)$. □

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