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Visualization of an entangled channel spin-1 system

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Covariance matrix formalism gives powerful entanglement criteria for continuous as well as finite dimensional systems. We use this formalism to study a mixed channel spin-1 system which is well known in nuclear reactions. A spin-j state can be visualized as being made up of $2j$ spinors which are represented by a constellation of $2j$ points on a Bloch sphere using Majorana construction. We extend this formalism to visualize an entangled mixed spin-1 system.

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Entanglement occurs as a result of quantum interference of states giving rise to nonclassical correlations between spatially separated quantum systems. Understanding and manipulating dynamics of these quantum features are of great importance for both fundamental physics and new emerging quantum technologies. Entanglement is needed in several quantum information processing tasks such as quantum cryptography [1], quantum teleportation [2], and quantum computation [3]. Characterization of entangled states is of great importance to understand the underlying mathematical structure of a given state. Entanglement of a bipartite system in a pure state is unambiguous and well defined. In contrast, mixed-state entanglement is relatively poorly understood. The purpose of this Brief Report is to visualize the notion of entanglement in statistical assemblies of particles with spin-1. There are several criteria [4] known to characterize entanglement like concurrence, entanglement of formation (EOF), positive partial transpose (PPT), etc. For a detailed discussion of this we refer the reader to the recent articles [4,5]. Here we employ covariance matrix formalism [6,7] as the elements of covariance matrix are closely related to intrinsic quantum correlations that exist between the constituent spinors of a spin system. Such correlations are shown to explain the physical origin of the squeezing behavior of the coupled spin-1 system [8,9]. It can be noted that the spin squeezing inequalities generalize the concept of spin squeezing parameter and provide the necessary and sufficient condition for genuine two qubit entanglement of spin-1 system [10]. Also an equivalence between the Peres-Horodecki criterion (PPT) and negativity of the covariance matrix has been established [6] showing that the condition is both necessary and sufficient for two qubit entanglement.

The correlations can be classified as those that arise (i) due to the coupling of the two subsystems and (ii) due to the projection of the combined total density matrix 'ρ_c' onto the desired spin space. In this Brief Report we have taken ρ_c to be a direct product of the two subsystem density matrices $\rho(1)$ and $\rho(2)$ which are not entangled and has no correlations of the first kind. However when we take the spin-1 projection of ρ_c, the correlation of second kind will appear.

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Our motivation here is to study entanglement of a channel spin-1 system which can be realized employing polarized spin $1/2$ beam and polarized spin $1/2$ target. Such systems naturally arise in nuclear physics experiments like hadron scattering and reaction processes [11–15]. This discussion is best done by employing the language of density matrix which can be applied with equal ease to pure as well as mixed spin systems. We outline the statistical tensor formalism using the well known spherical tensor representation for the density matrix and obtain the expressions for covariance matrix elements which represent spin-spin correlations between the constituent spinors. Further we go on to show that spin-1 density matrix can be visualized using Majorana construction [16].

It is well known that a symmetric two qubit state in which the two qubits are completely symmetric under interchange is confined to three-dimensional Hilbert-Schmidt space spanned by the eigenstates of the total angular momentum of the qubits $[|j=1,m\rangle; m=\pm 1, 0]$. It has been shown that [6] in the case of symmetric states the covariance matrix defined through

$$C_{ij} = [\langle \sigma_{1i}\sigma_{2j}\rangle - \langle \sigma_{1i}\rangle\langle \sigma_{2j}\rangle], \quad (1)$$

where $\sigma_{1i} = \sigma_i \otimes I, \sigma_{2i} = I \otimes \sigma_i$ (I is the 2×2 identity matrix and σ_i are the standard Pauli spin matrices), is necessarily positive semidefinite for separable symmetric state. It is also shown that negativity of the covariance matrix is a necessary and sufficient condition for entanglement in symmetric states.

The standard expression for the density matrix 'ρ' for a spin j system is

$$\rho = \frac{\mathrm{Tr}(\rho)}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} t_q^k \, \tau_q^{k^\dagger}, \quad (2)$$

where τ_q^k (with $\tau_0^0 = I$, the identity operator) are irreducible tensor operators of rank 'k' in the $2j+1$ dimension spin space with projection 'q' along the axis of quantization in the real three-dimensional space. The τ_q^k satisfy the orthogonality relations

$$\mathrm{Tr}\big(\tau_q^{k^\dagger}\tau_{q'}^{k'}\big) = (2j+1)\,\delta_{kk'}\delta_{qq'}. \quad (3)$$

Here the normalization has been chosen so as to be in agreement with Madison convention [17]. The spherical tensor parameters t_q^k which characterize the given system are the average expectation values given by

$$t_q^k = \frac{\mathrm{Tr}(\rho\, \tau_q^k)}{\mathrm{Tr}\rho}. \tag{4}$$

Since ρ is Hermitian and $\tau_q^{k^\dagger} = (-1)^q \tau_{-q}^k$, t_q^k satisfy the condition

$$t_q^{k^*} = (-1)^q\, t_{-q}^k. \tag{5}$$

Let us now consider the example of channel spin-1 system which plays an important role in nuclear reactions. A beam of nucleons colliding with a proton target provides such an example. If both the beam and the target are prepared to be in mixed states, then the corresponding density matrices are given by

$$\rho(i) = \frac{1}{2}[\, I + \vec{\sigma}(i)\cdot\vec{p}(i)\,] = \frac{1}{2}\sum_{k,q} t_q^k(i)\,\tau_q^{k^\dagger}(i); \quad i = 1, 2, \tag{6}$$

where $\vec{p}(i)$ are the polarization vectors and $\vec{\sigma}(i)'$s are the Pauli spin matrices.

The combined density matrix is the direct product of the individual matrices:

$$\rho_c = \rho(1) \otimes \rho(2). \tag{7}$$

Explicitly for spin-1 system [9],

$$t_q^1 = \left[\frac{\sqrt{6}}{3 + \vec{p}(1)\cdot\vec{p}(2)}\right][\vec{p}_q(1) + \vec{p}_q(2)], \tag{8}$$

$$t_q^2 = \left[\frac{2\sqrt{3}}{3 + \vec{p}(1)\cdot\vec{p}(2)}\right][\vec{p}(1)\otimes\vec{p}(2)]_q^2. \tag{9}$$

Let us consider the special Lakin frame (SLF) [9] which is widely used in studying nuclear reactions as follows: Choose $\hat{z}_0$ to be along $\vec{p}(1) + \vec{p}(2)$. Since $\vec{p}(1)$, $\vec{p}(2)$ together define a plane in any general situation, we choose $\hat{x}_0$ to be in this plane such that the azimuths of $\vec{p}(1)$, $\vec{p}(2)$ with respect to $\hat{x}_0$ are, respectively, 0 and π. The $\hat{y}_0$ axis is then chosen to be along $\hat{z}_0 \times \hat{x}_0$. The frame so chosen is indeed the special Lakin frame (SLF) as it is clear from the above Eqs. (8) and (9) that $t_{\pm 1}^1 = 0$ and $t_2^2 = t_{-2}^2$. In this frame, the Cartesian components of polarization vectors are given by

$$p_{x_0}(1) = \frac{p(1)p(2)\sin 2\theta}{|\vec{p}(1) + \vec{p}(2)|} = -p_{x_0}(2), \tag{10}$$

$$p_{y_0}(1) = p_{y_0}(2) = 0, \tag{11}$$

$$p_{z_0}(1) = \frac{p(1)^2 + p(1)p(2)\cos 2\theta}{|\vec{p}(1) + \vec{p}(2)|}; \quad p_{z_0}(2) = \frac{p(2)^2 + p(1)p(2)\cos 2\theta}{|\vec{p}(1) + \vec{p}(2)|}, \tag{12}$$

where 2θ is the angle between $\vec{p}(1)$ and $\vec{p}(2)$.

Choosing a simple case of $|\vec{p}(1)| = |\vec{p}(2)| = p$, we get $t_{\pm 1}^2 = 0$ in SLF. The density matrix so obtained is compared with that of the symmetric state density matrix given in equation (17) of Ref. [6]. It is observed that the covariance matrix takes the canonical form and the diagonal elements are given by

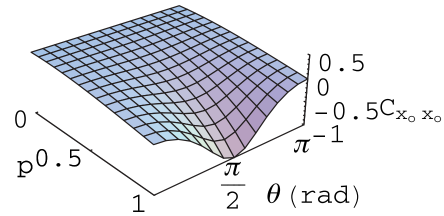

FIG. 1. (Color online) Variation of $C_{x_0x_0}$ as function of beam and target polarization p and θ(rad). $C_{x_0x_0} < 0$ indicates entanglement.

$$C_{x_0x_0} = \frac{1}{3} - \frac{\sqrt{2}}{3}t_0^2 + \frac{2}{\sqrt{3}}t_2^2 = \frac{1 - p^2(1 + 2\sin^2\theta)}{(3 + p^2\cos 2\theta)}, \tag{13}$$

$$C_{y_0y_0} = \frac{1}{3} - \frac{\sqrt{2}}{3}t_0^2 - \frac{2}{\sqrt{3}}t_2^2 = \frac{1 - p^2\cos 2\theta}{(3 + p^2\cos 2\theta)}, \tag{14}$$

$$C_{z_0z_0} = \frac{1}{3} + \frac{2\sqrt{2}}{3}t_0^2 - \left(\sqrt{\frac{2}{3}}\,t_0^1\right)^2 = \frac{1 + p^2(1 + 2\cos^2\theta)}{(3 + p^2\cos 2\theta)} - \left(\frac{4p\cos\theta}{(3 + p^2\cos 2\theta)}\right)^2. \tag{15}$$

Variation of $C_{x_0x_0}$ as function of θ (rad), beam and target polarization p is shown in Fig. 1. Variation of $C_{x_0x_0}$, $C_{y_0y_0}$, and $C_{z_0z_0}$ as a function of p for $\theta = \pi/4$ is shown in Fig. 2. Note that even if one of the diagonal elements is negative , the state is entangled. Also variation of $C_{x_0x_0}$ as function of θ for three values of p ($p = 0.5$, $p = 0.7$, and $p = 0.9$) is shown in Fig. 3. Range of θ for which the system is entangled is shown in Fig. 4. AOB, $A'OB'$ represent the region of entanglement for $\vec{p}(1)$, $\vec{p}(2)$, respectively, in SLF when $p = 0.9$. Similarly COD, $C'OD'$ represent the region of entanglement when $p = 0.7$. It is seen that as p increases range of entanglement also increases. In particular, when $p = 1$, i.e., when the states are pure, the

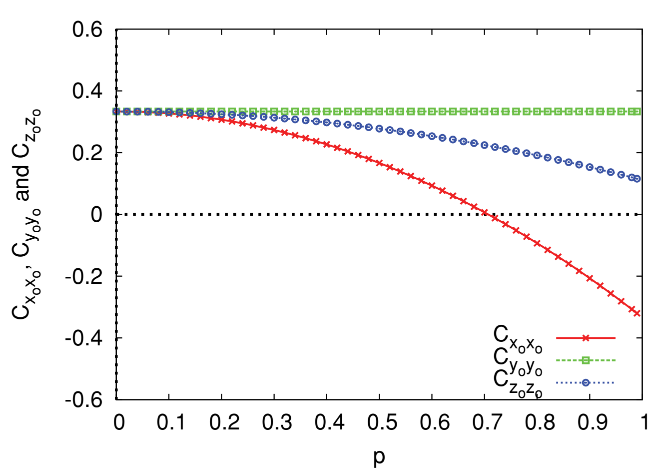

FIG. 2. (Color online) Variation of $C_{x_0x_0}$, $C_{y_0y_0}$, and $C_{z_0z_0}$ as a function of beam and target polarization p for $\theta = \pi/4$. $C_{x_0x_0} < 0$ indicates entanglement.

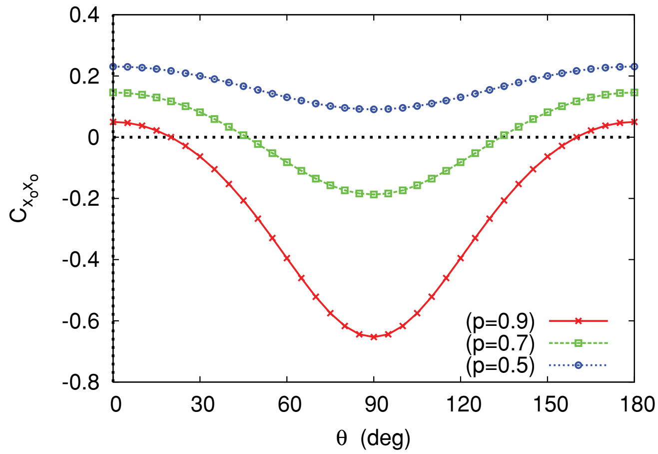

FIG. 3. (Color online) Variation of $C_{x_0x_0}$ as a function of angle θ (deg) for beam and target polarization $p = 0.5$, $p = 0.7$, and $p = 0.9$. $C_{x_0x_0} < 0$ indicates entanglement.

combined system will also be in a pure state and the spin-1 projection of this pure state will be in an entangled state except for $\theta = 0$ and π [equation (25) of Ref. [8]].

In order to visualize the entangled channel spin-1 system, let us discuss the geometric representation of the most general mixed spin-j density matrix. It has $(n^2 - 1)$ independent parameters where $n = (2j + 1)$ and is written as $\rho = \sum_{i=1}^{n} \lambda_i |\psi_i\rangle\langle\psi_i|$. Here $\lambda_i' s$ are the distinct eigenvalues belonging to $|\psi_i\rangle$. (This can be easily generalized to the degenerate case also.)

If all of these $|\psi_i\rangle$'s are the eigenkets of J_z, then the number of independent parameters gets reduced to $n + 1$ [$(n - 1)$ weights and (θ, ϕ) of the quantization axis]. Such a system is said to be oriented [18]. In all other cases at least one of the $|\psi_i\rangle$'s is not an eigenket of J_z. Such a system is said to be nonoriented spin system. In the most general nonoriented system none of the $|\psi_i\rangle$'s is an eigenket of J_z. Such a system can be parametrized using Majorana construction [16]

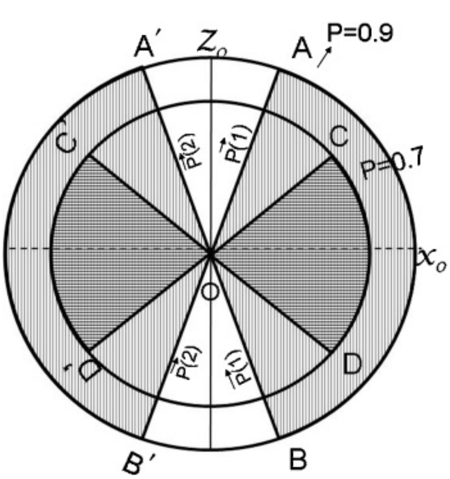

FIG. 4. Range of θ for which the system is entangled for beam and target polarization $p = 0.9$ (vertical lines) and $p = 0.7$ (horizontal lines). Extreme lines of the entangled regions represent the polarization vectors in SLF.

according to which a spin-j state can be written as

$$|\psi^j\rangle = \mathcal{N} \sum_{\mathcal{P}} \hat{\mathcal{P}}\{|\epsilon_1\epsilon_2 - - - \epsilon_{2J}\rangle\}, \tag{16}$$

$$\text{where,} \quad |\epsilon_r\rangle = \begin{pmatrix} \cos(\alpha_r/2) \\ \sin(\alpha_r/2)\, e^{i\beta_r} \end{pmatrix}$$

are the $2j$ spinors constituting the spin-j state. $\hat{\mathcal{P}}$ is a set of $N!$ permutations and $\mathcal{N}$ is the normalization factor. Since $|\psi^j\rangle$ can be expanded in terms of angular momentum basis vectors as $|\psi^j\rangle = \sum_{m=-j}^{+j} \mathcal{C}_m^j\, |jm\rangle$, the Majorana polynomial can be written as

$$P(Z) = \sum_{k=0}^{2j} (-1)^k \sqrt{\begin{pmatrix} k \\ 2j \end{pmatrix}}\; d_k Z^k, \tag{17}$$

where $k = j + m$ and $d_{j+m} = \mathcal{C}_m^j$ are the complex coefficients. Also $(k2j)$ represents the binomial coefficients and $Z = \tan(\alpha_r/2)\, e^{i\beta_r}$. These polynomials determine the orientation (α_r, β_r) of the constituent spinors [19], which can be visualized as a constellation of $2j$ points on the Bloch sphere.

Considering the particular example of channel spin-1 system in SLF, the eigenvalues λ_1, λ_2, and λ_3 of the density matrix are found to be

$$\lambda_{1,2} = \frac{1}{N}[1 + p^2\cos^2\theta \pm \sqrt{(p^4\sin^4\theta + 4p^2\cos^2\theta)}], \tag{18}$$

$$\lambda_3 = \frac{1 - p^2}{N}, \tag{19}$$

where $N = 3 + p^2\cos(2\theta)$. The corresponding eigenvectors, $|\psi_i\rangle; i = 1, 2$ and $|\psi_3\rangle$ are

$$|\psi_i\rangle = \frac{1}{N_i}\{p^2\sin^2\theta|11\rangle + [(1 + p\cos\theta)^2 - \lambda_i N]|1 - 1\rangle\} \tag{20}$$

$$|\psi_3\rangle = |10\rangle. \tag{21}$$

where $N_i = \{p^4\sin^4\theta + [(1 + p\cos\theta)^2 - \lambda_i N]^2\}^{1/2}$.

Substituting for d_{j+m}'s in Eq. (17), (α, β) of the two spinors for the state $|\psi_1\rangle$ are given by $(2\tan^{-1}\sqrt{|x|}, 0)$ and $(2\tan^{-1}\sqrt{|x|}, \pi)$ where $x = [(1 + p\cos\theta)^2 - \lambda_1 N]/p^2\sin^2\theta$. Observe that the spinors for the state $|\psi_1\rangle$ are confined to x_o-z_o plane. Similar construction for the state $|\psi_2\rangle$ indicates that the two spinors are in the y_o-z_o plane with (α, β) given by $(2\tan^{-1}\sqrt{|y|}, \pi/2)$and $(2\tan^{-1}\sqrt{|y|}, 3\pi/2)$ where $y = [(1 + p\cos\theta)^2 - \lambda_2 N]/p^2\sin^2\theta$. Since $|\psi_3\rangle$ is the $|10\rangle$ state, it is constituted by up and down spinors.

In this Brief Report we have studied the entanglement of a mixed spin-1 system as a function of polarization of spin 1/2 beam [$\vec{p}(1)$] and spin 1/2 target [$\vec{p}(2)$] using a covariance matrix formalism. In the particular case of $|\vec{p}(1)| = |\vec{p}(2)| = p$, it is found that as p increases the range of θ for which the system is entangled also increases where 2θ is the inclusive angle between $\vec{p}(1)$ and $\vec{p}(2)$. When the beam and the targets are in pure state, the resultant channel spin-1 system will also be in pure state and is entangled except for $\vec{p}(1)$ parallel and antiparallel to $\vec{p}(2)$. Further, using Majorana construction we have visualised channel spin-1 system as being made up of three sets of two spinors whose polar and azimuthal angles are functions of p and θ. It is found that two of the spinorial sets are confined to x_o-z_o plane and y_o-z_o plane of a Bloch sphere, respectively, and the remaining two are given by up and down spinors with respect to the z_o axis. Thus any entangled state can be visualized in terms of constituent spinors. Since none of the entanglement measures could capture the essence of mixed state entanglement, we believe that a deeper understanding of geometric characterisation of the Bloch sphere representation may lead to an intuitive understanding of entanglement. Our analysis can also be used to establish a quantitative relationship between spin-spin correlations and entanglement in higher spins. A detailed study of all these aspects using density matrix formalism is underway.

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