Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 13, Number 6 (2017), pp. 1669–1682

http://www.ripublication.com/gjpam.htm

A Two Variable Lucas Polynomials Corresponding to Hybrid Fibonacci Polynomials

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Abstract

Fibonacci numbers, Fibonacci polynomials in one variable, namely Catalan polynomials and Jacobsthal polynomials are recently generalized by the authors to two variables in the form of hybrid Fibonacci polynomials which exhibit many interesting combinatorial properties useful for research workers in combinatorics. In the present paper, two variable Lucas polynomials corresponding to two variable hybrid Fibonacci polynomials are defined and studied their combinatorial properties.

AMS subject classification: $05A19$, $30B70$ and $33C70$.
Keywords: Combinatorial identities, Fibonacci and Lucas numbers and hyper geometric functions in one or more variables.

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1. Introduction

Two Lucas polynomials denoted $l_n^{(C)}(x)$ and $l_n^{(J)}(x)$ are defined in the literature corresponding to Catalan polynomials and Jacobsthal polynomials denoted by $f_n^{(C)}(x)$ and $f_n^{(J)}(x)$ respectively which are natural extension of Fibonacci numbers F_n to one variable polynomials with many interesting combinatorial properties relevant to current literature [5, 7, 8, 9, 11, 12, 15]. The natural and the most beautiful three term recurrence relation is $F_{n+1} = F_n + F_{n-1}$, $F_1 = 1$, $F_2 = 1, n = 1, 2, 3, \ldots$ [1, 3, 4, 16] given by Fibonacci numbers. Closely connected to this is one more such relation $L_{n+1} = L_n + L_{n-1}$, $L_1 = 1$, $L_2 = 3$, $n = 1, 2, 3, \ldots$ given by Lucas numbers. Motivated by the relations, the sequence $l_n^{(C)}(x)$ is defined by the following three term recurrence relation [5]: $l_{n+1}^{(C)}(x) = xl_n^{(C)}(x) + l_{n-1}^{(C)}(x)$, with $l_1^{(C)}(x) = 1$, $l_2^{(C)}(x) = x$, $n = 1, 2, 3, \ldots$. Also the sequence $l_n^{(J)}(x)$ is defined by the following three term recurrence relation [5]: $l_{n+1}^{(J)}(x) = l_n^{(J)}(x) + xl_{n-1}^{(J)}(x)$, with $l_1^{(J)}(x) = 1$, $l_2^{(J)}(x) = 1 + 2x$, $n = 1, 2, 3, \ldots$.

The main aim of the paper is to define a two variable Lucas polynomial $l_n^{(H)}(x, y)$ corresponding to two variable hybrid Fibonacci polynomials [10] $f_n^{(H)}(x, y)$ which contain naturally both $l_n^{(C)}(x)$ and $l_n^{(J)}(x)$. The properties of both types of Lucas polynomials in one variable will be extended to hybrid polynomial in two variables. In the present paper, the two variable hybrid Lucas polynomials are shown to be directly connected to Tchebychev polynomials [6, 7, 8, 9, 11, 12, 14, 15] and the direct formula using Pascal like table is derived. The combinatorial properties such as generating function, matrix identities and determinant formula are derived.

2. Generalization of Lucas polynomials to two variables

Definition 2.1. The generalized hybrid polynomials in two variables x and y of degree n is denoted by $l_n^{(H)}(x, y)$ is

$$l_n^{(H)}(x, y) = \left[\left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^n + \left(\frac{x - \sqrt{x^2 + 4y}}{2}\right)^n\right]. \tag{2.1}$$

It is a two variable polynomial of degree n in x and y.

When $y = 1$ in (2.1), we get back Lucas-catalan polynomials in terms of x

$$l_n^{(H)}(x, y) = l_n^{(H)}(x, 1) = l_n^{(C)}(x).$$

When $x = 1$ in (2.1), the hybrid polynomials becomes Lucas-Jacobsthal polynomials in terms of y

$$l_n^{(H)}(x, y) = l_n^{(H)}(1, y) = l_n^{(J)}(y).$$

When $x = 1$ and $y = 1$ in (2.1), the hybrid polynomials becomes Lucas numbers

$$l_n^{(H)}(1, 1) = L_n.$$

Three Term Recurrence Relations

The hybrid polynomials satisfy the following three term recurrence relations:

$$l_{n+1}^{(H)}(x, y) = x\, l_n^{(H)}(x, y) + y\, l_{n-1}^{(H)}(x, y). \tag{2.2}$$

$$l_0^{(H)}(x, y) = 2,\ l_1^{(H)}(x, y) = x,\ n = 1, 2, 3, \dots .$$

The hybrid polynomials can also satisfies the following three term recurrence relation connected to hybrid Fibonacci polynomials [10] as follows:

$$l_n^{(H)}(x, y) = x\, f_n^{(H)}(x, y) + 2y\, f_{n-1}^{(H)}(x, y). \tag{2.3}$$

When $y = 1$ in (2.2), the three term recurrence relation computes the Lucas-catalan polynomial in terms of x

$$l_{n+1}^{(C)}(x) = x\, l_n^{(C)}(x) + l_{n-1}^{(C)}(x),$$

$$l_0^{(C)}(x) = 2,\ l_1^{C}(x) = x,\ n = 1, 2, 3, \dots .$$

When $x = 1$ in (2.2), the three term recurrence relation computes the Lucas-Jacobsthal polynomial in terms of y

$$l_{n+1}^{(J)}(y) = l_n^{(J)}(y) + y\, l_{n-1}^{(J)}(y),$$

$$l_0^{(J)}(y) = 2,\ l_1^{(J)}(y) = 1,\ n = 1, 2, 3, \dots .$$

When $x = 1$ and $y = 1$ in (2.2), the hybrid polynomials becomes the formula for Lucas numbers

$$L_{n+1} = L_n + L_{n-1},\ L_0 = 2,\ L_1 = 1,\ n = 1, 2, 3, \dots .$$

Initial Polynomials

The first eight initial polynomials of hybrid Lucas polynomials in two variable, Lucas-Catalan in terms of x and Lucas-Jacobsthal polynomial in terms of y is

n	$l_n^{(H)}(x, y)$	$l_n^{(C)}(x, 1)$	$l_n^{(J)}(1, y)$
0	2	2	2
1	x	x	1
2	$x^2 + 2y$	$x^2 + 2$	$1 + 2y$
3	$x^3 + 3xy$	$x^3 + 3x$	$1 + 3y$
4	$x^4 + 4x^2y + 2y^2$	$x^4 + 4x^2 + 2$	$1 + 4y + 2y^2$
5	$x^5 + 5x^3y + 5xy^2$	$x^5 + 5x^3 + 5x$	$1 + 5y + 5y^2$
6	$x^6 + 6x^4y + 9x^2y^2 + 2y^3$	$x^6 + 6x^4 + 9x^2 + 2$	$1 + 6y + 9y^2 + 2y^3$
7	$x^7 + 7x^5y + 14x^3y^2 + 7xy^3$	$x^7 + 7x^5 + 14x^3 + 7x$	$1 + 7y + 14y^2 + 7y^3$.

When $x = 1$ and $y = 1$, the hybrid polynomials becomes the sequence of Lucas numbers

$$L_n = 2, 1, 3, 4, 7, 11, 18, 29, \ldots .$$

Theorem 2.2. The explicit formula for hybrid polynomial in two variable is

$$l_n^{(H)}(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} y^k. \tag{2.4}$$

Proof. The result is proved by using mathematical induction method. For $n = 1$,

$$l_1^{(H)}(x, y) = \frac{1}{1-0} \binom{1-0}{0} x^{1-0} y^0 = x.$$

For $n = 2$,

$$l_2^{(H)}(x, y) = \frac{2}{2-0} \binom{2-0}{0} x^{2-0} y^0 + \frac{2}{2-1} \binom{2-1}{1} x^{2-2} y^1 = x^2 + 2y.$$

We assume the result is true for $n = m$. i.e.,

$$l_m^{(H)}(x, y) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{m-k} \binom{m-k}{k} x^{m-2k} y^k.$$

We prove the result is true for $n = m + 1$.

By applying the three term recurrence relation (2.2), for the hybrid polynomials

$$\begin{aligned}
l_{m+1}^{(H)}(x, y) =& x\, l_m^{(H)}(x, y) + y\, l_{m-1}^{(H)}(x, y) \\
=& x \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{m-k} \binom{m-1-k}{k} x^{m-2k} y^k \\
& + y \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \frac{m-k}{m-1-k} \binom{m-1-k}{k} x^{m-1-2k} y^k \\
=& x \Bigg[\frac{m}{m-0} \binom{m-0}{0} x^{m-0} y^0 + \frac{m}{m-1} \binom{m-1}{1} x^{m-2} y^1 \\
& + \frac{m}{m-2} \binom{m-2}{2} x^{m-4} y^2 \Bigg] + y \Bigg[\frac{m-1}{m-1} \binom{m-1}{0} x^{m-1} y^0 \\
& + \frac{m-1}{m-2} \binom{m-2}{1} x^{m-3} y^1 + \frac{m-1}{m-3} \binom{m-3}{2} x^{m-5} y^2 \Bigg] \\
=& \frac{m+1}{m-0} \binom{m-0}{0} x^m y^0 + \frac{m+1}{m-1} \binom{m-1}{1} x^{m-2} y \\
& + \frac{m+1}{m-2} \binom{m-2}{2} x^{m-4} y^2 + \cdots \\
=& \sum_{k=0}^{\lfloor \frac{m+1}{2} \rfloor} \frac{m+1}{m+1-k} \binom{m+1-k}{k} x^{m+1-2k} y^k
\end{aligned}$$

When $y = 1$ in (2.4), it becomes the explicit formula for the Lucas-Catalan polynomial in terms of x

$$l_n^{(H)}(x, y) = l_n^{(C)}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}.$$

When $x = 1$ in (2.4), it becomes the explicit formula for the Lucas-Jacobsthal polynomial in terms of y

$$l_n^{(H)}(x, y) = l_n^{(J)}(y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} y^k.$$

When $x = 1$ and $y = 1$ in (2.4), the hybrid polynomials becomes the explicit formula for Lucas numbers

$$L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k}.$$

■

Theorem 2.3. The hybrid polynomial can be connected to Tchebychev polynomials of first kind $T_n(x)$ and fourth kind $W_n(x)$ as follows:

1. $l_{2n}^{H}(x, y) = 2y^n \, T_n\left(1 + \frac{x^2}{2y}\right)$

2. $l_{2n+1}^{H}(x, y) = xy^n \, W_n\left(1 + \frac{x^2}{2y}\right)$

Proof.

(1) Consider

$$\begin{aligned}
l_{2n}^{(H)}(x, y) &= \left[\left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^{2n} - \left(\frac{x - \sqrt{x^2 + 4y}}{2}\right)^{2n}\right] \\
&= \left[\left(\left(1 + \frac{x^2}{2y}\right)y + y\sqrt{\left(1 + \frac{x^2}{2y}\right)^2 - 1}\right)^n \right. \\
&\quad \left. - \left(\left(1 + \frac{x^2}{2y}\right)y - y\sqrt{\left(1 + \frac{x^2}{2y}\right)^2 - 1}\right)^n\right] \\
&= y^n 2.\frac{1}{2}\left[\left(\left(1 + \frac{x^2}{2y}\right) + \sqrt{\left(1 + \frac{x^2}{2y}\right)^2 - 1}\right)^n \right. \\
&\quad \left. - \left(\left(1 + \frac{x^2}{2y}\right) - \sqrt{\left(1 + \frac{x^2}{2y}\right)^2 - 1}\right)^n\right] \\
&= 2y^n T_n\left(1 + \frac{x^2}{2y}\right)
\end{aligned}$$

(2) Consider

$$l_{2n+1}^{H}(x, y) = \left[\left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^{2n+1} + \left(\frac{x - \sqrt{x^2 + 4y}}{2}\right)^{2n+1}\right]$$

$$\begin{aligned}
&= \frac{x}{2}\left[\left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^{2n} + \left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^{2n}\right] \\
&\quad + \frac{\sqrt{x^2 + 4y}}{2}\left[\left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^{2n} - \left(\frac{x + \sqrt{x^2 + 4y}}{2}\right)^{2n}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{x.y^n}{2}\left[\left[\left(1+\frac{x^2}{2y}\right)+\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^n\right. \\
&\quad \left.+\left[\left(1+\frac{x^2}{2y}\right)-\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^n\right] \\
&\quad +\frac{\sqrt{x^2+4y}}{2}y^n\left[\left[\left(1+\frac{x^2}{2y}\right)+\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^n\right] \\
&- \frac{\sqrt{x^2+4y}}{2}y^n\left[\left[\left(1+\frac{x^2}{2y}\right)-\sqrt{\left(1+\frac{x^2}{2y}\right)^2-1}\right]^n\right] \\
&= xy^n\left[\left[2\left(1+\frac{x^2}{2y}\right)U_{n-1}\left(1+\frac{x^2}{2y}\right)-U_{n-2}\left(1+\frac{x^2}{2y}\right)\right]\right. \\
&\quad \left.+xy^nU_{n-1}\left(1+\frac{x^2}{2y}\right)\right] \\
&= xy^n\left[U_n\left(1+\frac{x^2}{2y}\right)+U_{n-1}\left(1+\frac{x^2}{2y}\right)\right] \\
&= xy^nW_n\left(1+\frac{x^2}{2y}\right)
\end{aligned}$$

■

When $y = 1$ in the Theorem 2.3, we deduce the following:

$$\begin{aligned}
l_{2n}^{(H)}(x, y) &= l_{2n}^{(C)}(x) = 2T_n\left(1+\frac{x^2}{2}\right) \\
l_{2n+1}^{(H)}(x, y) &= l_{2n+1}^{(c)}(x) = xW_n\left(1+\frac{x^2}{2}\right)
\end{aligned}$$

When $x = 1$ in the Theorem 2.3, we deduce the following:

$$l_{2n}^{(H)}(x, y) = l_{2n}^{(J)}(y) = 2y^nT_n\left(1+\frac{1}{2y}\right)$$

$$l_{2n+1}^{(H)}(x, y) = l_{2n+1}^{(J)}(y) = y^nW_n\left(1+\frac{1}{2y}\right)$$

An interesting special case

For $x = y = 1$ in the Theorem 2.3, we deduce the following:

$$\begin{aligned} l_{2n}^{(H)}(1,1) &= 2\,T_n\left(1+\frac{1}{2}\right) = 2\,T_n\left(\frac{3}{2}\right) \\ l_{2n+1}^{(H)}(1,1) &= W_n\left(1+\frac{1}{2}\right) = W_n\left(\frac{3}{2}\right) \end{aligned}$$

3. Combinatorial Properties of Hybrid Lucas Polynomials

In this section, the combinatorial properties such as generating function, matrix identities, and determinant formula are derived for the two variable hybrid Lucas polynomials.

Theorem 3.1. The generating function for generalized hybrid polynomials in two variable is

$$\sum_{n=0}^{\infty} l_n^{(H)}(x,y)t^n = \frac{2-xt}{1-tx-yt^2}.$$

Proof. Keeping in the mind the three term recurrence relation for $l_n^{(H)}(x,y)$, we proceed with the derivation. Put $l(x,y,t) = \sum_{n=0}^{\infty} l_n^{(H)}(x,y)t^n$.

We write

$$\begin{aligned} l(x,y,t) &= l_0^{(H)}(x,y) + l_1^{(H)}(x,y)t + \cdots + l_{n+1}^{(H)}(x,y)t^{n+1} + \cdots \\ -xtl(x,y,t) &= -xl_0^{(H)}(x,y)t - xl_1^{(H)}(x,y)t^2 - \cdots - xl_n^{(H)}(x,y)t^{n+1} - \cdots \\ -yt^2l(x,y,t) &= -l_0^{(H)}(x,y)yt^2 - l_1^{(H)}(x,y)yt^3 - \cdots + l_{n-1}^{(H)}(x,y)yt^{n+1} - \cdots. \end{aligned}$$

Summing all the three expressions on both sides, we get

$$\begin{aligned} (1-tx-yt^2)l(x,y,t) &= 2 + xt - tx(2) - yt^2.2 + (x^2+2y)t^2 + \cdots \\ l(x,y,t) &= \frac{2-xt}{1-tx-yt^2} \end{aligned}$$

■

Put $y = 1$, in the above Theorem 3.1, it becomes generating function for Lucas-Catalan polynomials in terms of x

$$\sum_{n=0}^{\infty} l_n^{(C)}(x)t^n = \frac{2-xt}{1-tx-t^2}.$$

Put $x = 1$, in the above Theorem 3.1, it becomes generating function for Lucas-Jacobsthal polynomials in terms of y

$$\sum_{n=0}^{\infty} l_n^{(J)}(y)t^n = \frac{2-t}{1-t-yt^2}.$$

When $x = 1$ and $y = 1$ in the above Theorem 3.1, the generating function for hybrid polynomials becomes generating function of Lucas numbers

$$\sum_{n=0}^{\infty} L_n t^n = \frac{2-t}{1-t-t^2}.$$

Theorem 3.2. The generalized polynomials in two variable can be expressed in matrix form in odd and even functions are as follows:

(1) $$\begin{bmatrix} l_{2n+4}^{(H)}(x,y) & l_{2n+2}^{(H)}(x,y) \\ l_{2n+2}^{(H)}(x,y) & l_{2n}^{(H)}(x,y) \end{bmatrix} = \begin{bmatrix} l_4^{(H)}(x,y) & l_2^{(H)}(x,y) \\ l_2^{(H)}(x,y) & l_0^{(H)}(x,y) \end{bmatrix} \cdot \begin{bmatrix} x^2+2y & 1 \\ -y^2 & 0 \end{bmatrix}^n$$

(2) $$\begin{bmatrix} l_{2n+5}^{(H)}(x,y) & l_{2n+3}^{(H)}(x,y) \\ l_{2n+3}^{(H)}(x,y) & l_{2n+1}^{(H)}(x,y) \end{bmatrix} = \begin{bmatrix} l_5^{(H)}(x,y) & l_3^{(H)}(x,y) \\ l_3^{(H)}(x,y) & l_1^{(H)}(x,y) \end{bmatrix} \cdot \begin{bmatrix} x^2+2y & 1 \\ -y^2 & 0 \end{bmatrix}^n$$

Proof. (1) The theorem is proved by using the principle of mathematical induction:
For $n = 1$ the result is true.

$$\begin{bmatrix} l_6^H(x,y) & l_4^H(x,y) \\ l_4^H(x,y) & l_2^H(x,y) \end{bmatrix} = \begin{bmatrix} l_4^H(x,y) & l_2^H(x,y) \\ l_2^H(x,y) & l_0^H(x,y) \end{bmatrix} \cdot \begin{bmatrix} x^2+2y & 1 \\ -y^2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} l_6^H(x,y) & l_4^H(x,y) \\ l_4^H(x,y) & l_2^H(x,y) \end{bmatrix}.$$

We assume the result is true for $n = k$.

$$\begin{bmatrix} l_{2k+4}^{(H)}(x,y) & l_{2k+2}^{(H)}(x,y) \\ l_{2k+2}^{(H)}(x,y) & l_{2k}^{(H)}(x,y) \end{bmatrix} = \begin{bmatrix} l_4^{(H)}(x,y) & l_2^{(H)}(x,y) \\ l_2^{(H)}(x,y) & l_0^{(H)}(x,y) \end{bmatrix} \cdot \begin{bmatrix} x^2+2y & 1 \\ -y^2 & 0 \end{bmatrix}^k .$$

Now we prove the result is true for $n = k + 1$.

Consider

$$\begin{bmatrix} l_{2k+4}^{(H)}(x,y) & l_{2k+2}^{(H)}(x,y) \\ l_{2k+2}^{(H)}(x,y) & l_{2k}^{(H)}(x,y) \end{bmatrix} \cdot \begin{bmatrix} x^2+2y & 1 \\ -y^2 & 0 \end{bmatrix} = \begin{bmatrix} l_{4}^{(H)}(x,y) & l_{2}^{(H)}(x,y) \\ l_{2}^{(H)}(x,y) & l_{0}^{(H)}(x,y) \end{bmatrix}$$

$$\times \begin{bmatrix} x^2+2y & 1 \\ -y^2 & 0 \end{bmatrix}^k \cdot \begin{bmatrix} x^2+2y & 1 \\ -y^2 & 0 \end{bmatrix}.$$

On simplification by using the three term recurrence relation for hybrid polynomials we get

$$\begin{bmatrix} l_{2k+6}^{(H)}(x,y) & l_{2k+4}^{(H)}(x,y) \\ l_{2k+4}^{(H)}(x,y) & l_{2k+2}^{(H)}(x,y) \end{bmatrix} = \begin{bmatrix} l_{4}^{(H)}(x,y) & l_{2}^{(H)}(x,y) \\ l_{2}^{(H)}(x,y) & l_{0}^{(H)}(x,y) \end{bmatrix} \cdot \begin{bmatrix} x^2+2y & 1 \\ -y^2 & 0 \end{bmatrix}^{k+1}$$

The proof of 2 is similar to that of 1. ■

For $y = 1$, in the Theorem 3.2, we obtain matrix identities for Lucas-Catalan polynomials in the following form:

$$(3) \quad \begin{bmatrix} l_{2n+4}^{(C)}(x) & l_{2n+2}^{(C)}(x) \\ l_{2n+2}^{(C)}(x) & l_{2n}^{(C)}(x) \end{bmatrix} = \begin{bmatrix} l_{4}^{(C)}(x) & l_{2}^{(C)}(x) \\ l_{2}^{(C)}(x) & l_{0}^{(C)}(x) \end{bmatrix} \cdot \begin{bmatrix} x^2+2 & 1 \\ -1 & 0 \end{bmatrix}^n$$

$$(4) \quad \begin{bmatrix} l_{2n+5}^{(C)}(x) & l_{2n+3}^{(C)}(x) \\ l_{2n+3}^{(C)}(x) & l_{2n+1}^{(C)}(x) \end{bmatrix} = \begin{bmatrix} l_{5}^{(C)}(x) & l_{3}^{(C)}(x) \\ l_{3}^{(C)}(x) & l_{1}^{(C)}(x) \end{bmatrix} \cdot \begin{bmatrix} x^2+2 & 1 \\ -1 & 0 \end{bmatrix}^n$$

For $x = 1$ in the Theorem 3.2, we obtain matrix identities for Lucas-Jacobsthal polynomials in the following form:

$$(5) \quad \begin{bmatrix} l_{2n+4}^{(J)}(y) & l_{2n+2}^{(J)}(y) \\ l_{2n+2}^{(J)}(y) & l_{2n}^{(J)}(y) \end{bmatrix} = \begin{bmatrix} l_{4}^{(J)}(y) & l_{2}^{(J)}(y) \\ l_{2}^{(J)}(y) & l_{0}^{(J)}(y) \end{bmatrix} \cdot \begin{bmatrix} 1+2y & 1 \\ -1 & 0 \end{bmatrix}^n$$

$$(6) \quad \begin{bmatrix} l_{2n+5}^{(J)}(y) & l_{2n+3}^{(J)}(y) \\ l_{2n+3}^{(J)}(y) & l_{2n+1}^{(J)}(y) \end{bmatrix} = \begin{bmatrix} l_{5}^{(J)}(y) & l_{3}^{(J)}(y) \\ l_{3}^{(J)}(y) & l_{1}^{(J)}(y) \end{bmatrix} \cdot \begin{bmatrix} 1+2y & 1 \\ -1 & 0 \end{bmatrix}^n$$

For $x = 1$ and $y = 1$ in the Theorem 3.2, we obtain Lucas matrix identities as follows:

$$(7)\quad \begin{bmatrix} L_{2n+4} & L_{2n+2} \\ L_{2n+2} & L_{2n} \end{bmatrix} = \begin{bmatrix} L_4 & L_2 \\ L_2 & L_0 \end{bmatrix} . \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}^n$$

$$(8)\quad \begin{bmatrix} L_{2n+5} & L_{2n+3} \\ L_{2n+3} & L_{2n+1} \end{bmatrix} = \begin{bmatrix} L_5 & L_3 \\ L_3 & L_1 \end{bmatrix} . \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}^n$$

Determinants Formulas

A direct application of the following three term recurrence relation

$$(i)\quad [l_{2n}^{(H)}(x, y) - 2y^n] = (x^2 + 2y)[l_{2n-2}^{(H)}(x, y) - 2y^{n-1}] - y[l_{2n-4}^{(H)}(x, y) - 2y^{n-2}]$$

$$(ii)\quad x\, l_{2n+1}^{(H)}(x, y) = [l_{2n+2}^{(H)}(x, y) - 2y^{n+1}] - y[l_{2n}^{(H)}(x, y) - 2y^n]$$

will yield the two determinant formulas stated in the following theorem with out proof.

Theorem 3.3. The determinants formulas for generalized hybrid polynomials are

$$l_{2n}^{H}(x, y) - 2y^n = \begin{vmatrix} x^2+2y & -y & 0 & \cdots & 0 & -y \\ -y & x^2+2y & -y & 0 & \cdots & 0 \\ 0 & -y & x^2+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -y & 0 & 0 & 0 & -y & x^2+2y \end{vmatrix}_{nXn}$$

$$x l_{2n+1}^{(H)}(x, y) = \begin{vmatrix} x^2+2y & -y & 0 & \cdots & 0 & -y \\ -y & x^2+2y & -y & 0 & \cdots & 0 \\ 0 & -y & x^2+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -y & 0 & 0 & 0 & -y & x^2+2y \end{vmatrix}_{(n+1)X(n+1)}$$

$$- \quad y \begin{vmatrix} x^2+2y & -y & 0 & \cdots & 0 & -y \\ -y & x^2+2y & -y & 0 & \cdots & 0 \\ 0 & -y & x^2+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -y & 0 & 0 & 0 & -y & x^2+2y \end{vmatrix}_{nXn}$$

for $n = 3, 4, 5, \cdots$

Special cases

(1) For $y = 1$ and $n = 3, 4, 5, \cdots$ we deduce

$$l_{2n}^{(C)}(x) - 2 = \begin{vmatrix} x^2+2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & x^2+2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & x^2+2 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & -1 & x^2+2 \end{vmatrix}_{nXn}$$

$$x l_{2n+1}^{(C)}(x) = \begin{vmatrix} x^2+2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & x^2+2 & -1 & 0 & \cdots & 0-1 \\ 0 & -y & x^2+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & -1 & x^2+2 \end{vmatrix}_{(n+1)X(n+1)}$$

$$- \begin{vmatrix} x^2+2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & x^2+2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & x^2+2 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & -1 & x^2+2 \end{vmatrix}_{nXn}$$

(2) For $x = 1$ and $n = 3, 4, 5, \cdots$ we deduce

$$l_{2n}^{(J)}(x, y) - 2y^n = \begin{vmatrix} 1+2y & -y & 0 & \cdots & 0 & -y \\ -y & 1+2y & -y & 0 & \cdots & 0 \\ 0 & -y & 1+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -y & 0 & 0 & 0 & -y & 1+2y \end{vmatrix}_{nXn}$$

$$l_{2n+1}^{(C)}(y) = \begin{vmatrix} 1+2y & -y & 0 & \cdots & 0 & -y \\ -y & 1+2y & -y & 0 & \cdots & 0 \\ 0 & -y & 1+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -y & 0 & 0 & 0 & -y & 1+2y \end{vmatrix}_{(n+1)X(n+1)}$$

$$- \quad y \begin{vmatrix} 1+2y & -y & 0 & \cdots & 0 & -y \\ -y & 1+2y & -y & 0 & \cdots & 0 \\ 0 & -y & 1+2y & \ddots & -y & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -y & 0 & 0 & 0 & -y & 1+2y \end{vmatrix}_{nXn}$$

(3) For $x = 1,\ y = 1$ and $n = 3, 4, 5, \cdots$ we deduce

$$L_{2n} - 2 = \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & -1 & 3 \end{vmatrix}_{nXn}$$

$$\begin{aligned} L_{2n+1} &= \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ -1 & -1 & 3 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & -1 & 3 \end{vmatrix}_{(n+1)X(n+1)} \\ &- \begin{vmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & \ddots & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & -1 & 3 \end{vmatrix}_{nXn} \end{aligned}$$

Acknowledgement

The second author would like to thank both UGC-SWRO, F. No. FIP/12th Plan/KADA018 TF02 and Govt. of Karnataka (DCE) and the third author would like to thank UGC, Govt. of India for encouraging this work under Post Doctoral Fellowship For SC/ST Candidates Order No. F./PDFSS - 2014 -15-ST-KAR-10116.

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