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Independent Domination Critical and Stable Graphs Upon Edge Subdivision

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ABSTRACT

A subset D of vertices in a graph G is an independent dominating set if D is a dominating set and an independent set. A graph is independent domination edge subdivision critical if the subdivision of an arbitrary edge increases the independent domination number. On the other hand, a graph is independent domination edge subdivision stable if the subdivision of an arbitrary edge leaves the independent domination number unchanged. In this paper, we continue the study of independent domination critical and stable graphs upon edge subdivision.

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1. INTRODUCTION

Let G=(V, E) be a graph with vertex set V and edge set E. A subset D of V is an independent dominating set if D is an independent set and every vertex of Vis adjacent to some vertex in D. Independent domination number denoted i(G) is the minimum cardinality of an independent dominating set. An independent dominating set of cardinality i(G) is called an i(G)-set. When a graphical parameter is of interest in an application, it is not only important to study the parameter but also it is important to know how the parameter behaves when the graph is modified. For instance, the effects of removing or adding an edge, or removing a vertex have been considered on the parameter domination number. In this paper we consider the graph parameter independent domination number and edge subdivision as the graph modification.

Sumner *et al.*[8] initiated the study of domination critical graphs. That is graphs whose domination number decreases when an edge in the complement of the graph is added.

Removal of a vertex can increase the domination number by more than one, but can decrease it by at most one. Motivated by this Brigham[2] defined the concept of vertex domination critical graph. Van der Merwe[9] initiated the study of those graphs where the total domination number decreases upon the addition of any edge. Lemanska *et al.*[5,6] discussed the concept of weakly connected domination critical graphs and characterized the trees which are weakly connected domination stable. Ao[1] considered the effect of addition of edge on the independent domination number and initiated the study of independent domination critical graphs. Edward *et al.*[3] introduced the independent domination vertex critical graphs. Yamuna *et al.*[10] introduced the concept domination stable and critical graph upon edge subdivision. Sharada *et al.*[7] introduced the concept of independent domination critical and stable graphs upon edge subdivision.

We continue the study of independent domination critical and stable graphs. We also characterize the trees which are independent domination critical. An edge *uv* is said to be subdivided if the edge *uv* is deleted and two new edges *ux* and *xv* are added. *x* is a new vertex and is called the subdivision vertex. Subdivision of an edge in a graph can cause its independent domination number to increase, to decrease, or to remain the same. Let G_e^* denote the graph obtained on subdividing an arbitrary edge e of G.

All graphs considered in this paper are finite and simple. For definitions not given here and notations not presented see[4]. A tree is an acyclic graph. A leaf of a tree is a vertex of degree 1. A support vertex is a vertex adjacent to a leaf. A strong support vertex is a support vertex that is adjacent to more than one leaf.

2. KNOWN DEFINITIONS

We have the following weak partition of an edge set $E(G)$, where by a weak partition of a set we mean a partition of the set in which some of the subsets may be empty.

Definition 1 [7]: For a graph G, a weak partition of its edge set is given by $E(G) = E^{\circ}(G) \cup E^{-}(G) \cup E^{+}(G)$.

Definition 2 [7]: A graph G is defined to be independent domination edge subdivision critical, or $i - critical\ for\ short, if\ i(G_e^*) > i(G)\ for\ every\ edge\ e \in E(G)$. In other words, $G\ is\ i - critical\ if\ E(G) = E^{+}(G)$.

Definition 3 [7]: A graph $G\ is\ i - stable\ if\ i(G_e^*) = i(G)\ for\ every\ edge\ e \in E(G)$. In other words, G is $i - stable\ if\ E(G) = E^{\circ}(G)$.

Definition 4 [7]: A graph $G\ is\ i - changing\ if\ i(G_e^*) \neq i(G)$ for every edge $e \in E(G)$. Thus a graph G is *i*-changing if the subdivision of any edge from G either increases or decreases the independent domination number, that is, $E(G) = E^{-}(G) \cup E^{+}(G)$. It follows that *i*-critical graphs are a subset of *i*-changing graphs.

3. MAIN RESULTS

Lemma 1. $If\ e$ is an edge of a graph G then $i(G_e^*) \leq\ i(G)\ +\ 1$.

Proof. Let $e\ =\ uv$ be an edge of G. Let uv be subdivided to form two new edges us and sv, where s is the new vertex. Let D be a minimum dominating set of G.
We consider two cases.

Case 1: u, v does not belong to D. Then $D\ \cup\ \{s\}$ is an independent dominating set of G_e^*. Thus, $i(G_e^*) \leq\ |D\ \cup\ \{s\}\ =\ i(G)\ +\ 1$.

Case 2: $|u, v\ \cap\ D| =\ 1$. Without loss of generality suppose $u\ \in\ D$ and v' does not belong to D. Then $D\ \cup\ \{v\}$ is an independent dominating set of G_e^*. Thus, $i(G_e^*\ ) \leq\ |D\ \cup\ \{v\}|$
$=\ i(G)\ +\ 1$.

Observation 2. $Let\ G\ be\ i - critical\ then\ i(\ G_e^*) =\ i(G)\ +\ 1, for\ every\ e\ \in\ E(G).$

Lemma 3. $If\ G\ is\ i - critical$, then no two support vertices are adjacent.

Proof. Suppose that u and v are adjacent support vertices adjacent to pendant vertices u' and v' respectively. Let G^* be the graph obtained on subdividing the edge uv of G. Let s be the new subdivision vertex. Let D' be a minimum dominating set in G'. We consider two cases.

Case 1: $u, v\ \in\ D'$

Then $D_1 =\ (D' \setminus \{u, v\})\ \cup\ \{u', v'\}\ or\ D_2\ =\ (D' \setminus \{u\})\ \cup\ \{u'\}$ is an independent dominating set of G and $i(G) \leq\ |D_1| = |D_2|\ =\ i(G^*)$, a contradiction, since $|D| = |D'|$ and G is $i - critical.$

Case 2: $|\{u, v\} \cap\ D'| = 1$

Without loss of generality, suppose $u\ \in D'$ and v does not belong to D'. Then D' is an independent dominating set of G and $i(G)\ \leq |D'| = i(G^*)$, a contradiction.

Lemma 4. $If\ G$ is $i - critical$ then for every two supports u, v there is $d(u, v)\ \geq\ 3$.

Proof. By previous Lemma, there is $d(u, v) > 1$ for every two supports u, v. Suppose that u and v are support vertices in an $i - critical$ graph G and $d(u, v)\ =\ 2$. Let w be the intermediate vertex joining u and v. Subdivide uw to form two new edges us and sw, where S is the new vertex. Let G^* be the graph obtained on subdividing uw. Without loss of generality we assume that $u, v\ \in\ D'$ and s does not belong to D'. Then D' is also an independent dominating set of G. Hence, $i(G) \leq |D'| = i(G^*)$, a contradiction, since G is $i - critical.$

Lemma 5. $Let\ G\ be\ i - stable\ then$

1. Every support vertex has exactly one pendant vertex adjacent to it.
2. If v is a pendant vertex then there exists an $i(G)$-set containing v.

Proof. 1. Let u be a support vertex adjacent to two pendant vertices x, y. Then u is contained in every $i(G) - set$. Let $e = ux$ and $e' = uy$, then $i(G_e^*) = i(G_{e'}^*) = i(G) + 1$, which is a contradiction as G is i-stable.

2. Let u be a support vertex adjacent to the pendant vertex v. Subdivide uv to form two new edges us and sv, where s is the subdivision vertex. Let $e = uv$ and $D\ be\ an\ i(G_e^*) - set$. We have $v \in D\ orv$ does not belong to D. If $v \in D$ then D is also an $i(G) - set$. If v does not belong to D then s is contained in $i(G_e^*) - set$, since v is pendant. Hence $(D \setminus \{s\}) \cup \{v\}$ is an $i(G) - set$. Thus in both the cases, v belongs to some $i(G)$-set.

Lemma 6. Every graph is an induced sub graph of an *i-stable* graph.

Proof. Let G be a graph with n vertices say $u_1, u_2, \ldots, u_n$. Let $H = G \circ K_1$. Label the pendant vertices of $H\ as\ v_1, v_2, \ldots, v_n$. Clearly $\{v_1, v_2, \ldots, v_n\}$ is an $i(H)$-set. Let H_{ei}^* denote the graph obtained on subdividing the edge e_i of H, where $e_i = (u_i, v_i)$, for $i = 1, 2, \ldots, n$. Let s_i denote the new vertex obtained on subdividing e_i for $i = 1, 2, \ldots, n$. Then $(\{v_1, v_2, \ldots, v_n\} \setminus \{v_i\}) \cup \{s_i\}$ is an $i(H_{ei}^*)$-set. Consider H_{ej}^*,where $e_j = (u_j, u_j + 1)$, for $j = 1, 2, \ldots, n$. Then $(\{v_1, v_2, \ldots, v_n\} \setminus \{v_j + 1\}) \cup \{u_j + 1\}$ is an $i(H_{ej}^*)$-set. In both the cases $i(H_{ei}^*) = i(H)$. Therefore H is $i - stable$.

Independent Domination Critical Trees

We are now in a position to constructively characterize all i-critical trees. For this purpose, we define a family of trees and two operations. $If\ T$ is a tree, then we define the status of a vertex v, denoted $sta(v)$, to be A or B. Let F be a family of trees that can be obtained from a sequence $T_1, \ldots, T_j$, $(j > 1)$ of trees such that T_1 is a star $K_{1,m}$ for $m > 1$, where initially $sta(v) = A$ for the support vertex v of T_1, $sta(u) = B$ for every leaf u of T_1 and $T = T_j$, and, if $j > 2$, then T_{i+1} can be obtained from T_i by one of the operations listed below. Once a vertex is assigned a status, this status remains unchanged as the tree is recursively constructed.

Operation O₁: The tree T_{i+1} is obtained from T_i by adding a path u, v, w and an edge wy, where y is a vertex of T_i such that $sta(y) = B$. Let $sta(u) = B$, $sta(v) = A$ and $sta(w) = B$.

Operation O₂: The tree T_{i+1} is obtained from T_i by adding a path u, v, w, x and an $edge\ xy$, where y is a vertex of T_i such that $sta(y) = A$. Let $sta(u) = B, sta(v) = A, sta(w) = B$ and $sta(x) = B$.

Lemma 7. $If\ T \in F$, then there is the unique minimum independent dominating set of T.

Proof. Let $T \in F$. Then every vertex of T is assigned with a status. Assume there are k vertices with status A in T. Then $D = \{a_1, \ldots, a_k\}$, where $sta(a_i) = A$ for $i = 1, \ldots, k$ is the unique minimum independent dominating set of T.

Lemma 8. $If\ T$ is a tree with at least three vertices and D is the unique minimum independent dominating set in T, then D contains no leaves.

Proof. Suppose there is a leaf v in D, where D is the unique minimum independent dominating set of T. Then $(D \setminus \{v\}) \cup \{u\}$, where u is the only neighbor of v, is a minimum independent dominating set of T, a contradiction.

Theorem 9. $If\ T$ is a tree with at least three vertices, then $T \in F$ if and only if there is a unique minimum independent dominating set in T.

Proof. $If\ T \in F$, then the result follows from Lemma 7.

Let T be a tree with at least three vertices and assume there is the unique minimum independent dominating set in T. We use induction on $i(T)$, the independent domination number of T. If $i(T) = 1$, then T is a star with at least two leaves and of course $T \in F$. Assume that $i(T) > 1$ and let $P = (v_0, \ldots, v_l)$ be a longest path in T. Since $i(T) > 1$, we have $l > 2$. Let D be the unique minimum independent dominating set in T. From Lemma 2 we have $v0$ does not belong to D. Thus $v_1 \in D$. We now consider two possibilities depending on the degree of v_3.

Case 1: $deg(v_3) > 2$

Define $T' = T \setminus \{v_0, v_1, v_2\}$. It is possible to observe that there is a unique minimum independent dominating set in T' and $i(T') < i(T)$. Thus by the induction hypothesis, $T' \in F$. If v_3 is a support vertex then $v_3 \in D$. Moreover if $deg(v_1) = 2$ in T, then $(D \setminus \{v_1\}) \cup \{v_0\}$ would be another $i(T)$ −set, which is a contradiction. Hence v_3 is not a support vertex. Thus $sta(v_3) = B$ in T'. Thus T is obtained from T' by operation O_1.

Case 2: $deg(v_3) = 2$

Define $T' = T \setminus \{v_0, v_1, v_2, v_3\}$. It is possible to observe that there is a unique minimum independent dominating set in T' and $i(T') < i(T)$. Thus by the induction hypothesis, $T' \in F$. Suppose v_4 is not a support vertex in T', then v_4 does not belong to D. Without loss of generality we assume that $v_3 \in D$. As in case 1 we can prove that D is not the unique minimum independent dominating set, this is a contradiction. Hence v_4 is a support vertex in T'. Thus $sta(v_4) = A$ in T'. Thus T is obtained from T' by operation O_2.

Theorem 10. A tree T is $i - critical$ if and only if there is a unique minimum independent dominating set in T.

Proof. Let T be a tree. Suppose there is a unique minimum independent dominating set in T and T is not $i - critical$. Then there is an edge such that $i(T_e^*) = i(T)$, where T_e^* is obtained by subdividing the edge e of T. Let D' be a minimum independent dominating set of T_{e*}. Let s be the new vertex obtained on subdividing the edge $e = uv$. We consider three possibilities.

Case 1: If u, v does not belong to D' then s must belong to D'. Then $D_1 = (D' \setminus \{s\}) \cup \{u\}$ and $D_2 = (D' \setminus \{s\}) \cup \{v\}$ are independent dominating sets in T and $|D_1| = |D_2| = i(T_e^*) = i(T)$, which is a contradiction to the fact that there exists exactly one minimum independent dominating set in T.

Case 2: If u, v belong to D', let $u' \neq u$ be a neighbor of u and $v' \neq v$ be a neighbor of v. Then $D_1 = (D' \setminus \{u\}) \cup \{u'\}$ and $D_2 = (D' \setminus \{v\}) \cup \{v'\}$ are independent dominating sets in T and $|D_1| = |D_2| = i(T_e^*) = i(T)$, which is a contradiction to the fact that there exists exactly one minimum independent dominating set in T.

Case 3: Exactly one of the vertices of u, v belongs to D'. Assume that u belongs to D' and v does not belong to D'. If s belongs to D' then define $D_1 = (D' \setminus \{s\}) \cup \{v\}$ and if s does not belong to D', define $D_1 = (D' \setminus \{v'\}) \cup \{v\}$, where v' is a neighbor of v. Then D_1 and $D_2 = D'$ are independent dominating sets in T and $|D_1| = |D_2| = i(T_e^*) = i(T)$, which is a contradiction to the fact that there exists exactly one minimum independent dominating set in T.

Now we show that if T is $i - critical$, then there exists exactly one minimum independent dominating set in T. Suppose to the contrary that there are at least two $i(T)$-sets, say D_1 and D_2. Then $|D_1 \oplus D_2| > 2$, where $(D_1 \oplus D_2) = (D_1 \setminus D_2) \cup (D_2 \setminus D_1)$.

Claim 1: Every vertex belonging to $D_1 \setminus D_2$ has a neighbor in $D_2 \setminus D_1$ and every vertex belonging to $D_2 \setminus D_1$ has a neighbor in $D_1 \setminus D_2$. Suppose this is not true, let $u \in (D_1 \setminus D_2)$ and $N_T(u) \cap (D_2 \setminus D_1) = \emptyset$. Then of course u does not belong to D_2. But we observe that every neighbor of u belongs to D_2. Since $N_T(u) \cap (D_2 \setminus D_1) = \emptyset$, we have $N_T(u) \subseteq D_1$. But then $D_1 \setminus \{u\}$ is a smaller independent dominating set of T, which gives a contradiction.

Since T is a tree, Claim 1 implies that $T(D_1 \oplus D_2)$ is a non-trivial forest. Let u be a leaf of $T(D_1 \oplus D_2)$. Without loss of generality let $u \in (D_1 \setminus D2)$ and let v be the neighbor of u such that $v\ 2\ (D_2 \setminus D_1)$. Let us choose v such that v is not a leaf of T (if v is a leaf of T, then we can take u instead of v and v instead of u).

Let uv be subdivided to form us and sv, where s is the subdivision vertex. Then $D = (D1 \setminus \{u\})\ [\ \{s\}$ is an independent dominating set of T_e^* and $i(T_e^*) = |D| = |D_1| = i(T)$, which contradicts the fact that T is $i - critical$.

Corollary 11. Let T be a tree of order at least three. Then the following conditions are equivalent:

- T belongs to the family $F;$
- T is $i - critical$;
- There is exactly one minimum independent dominating set in T;

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