The Sum-Eccentricity Energy Of A Graph

Mohammad Issa Sowaity and B. Sharada

Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru - 570 006, INDIA
E-mail address: mohammad_d2007@hotmail.com

Department of Studies in Computer Science, University of Mysore, Manasagangotri, Mysuru - 570 006, INDIA
E-mail address: sharadab21@gmail.com

Abstract. In this paper, we introduce the concept of the sum-eccentricity matrix $S_e(G)$ of a graph $G$ and obtain some coefficients of the characteristic polynomial $P(G, \lambda)$ of the sum-eccentricity matrix of $G$. We also introduce the sum-eccentricity energy $ES_e(G)$ of a graph $G$. Sum-eccentricity energies of some well-known graphs are obtained. Upper and lower bounds for $ES_e(G)$ are established. It is shown that if the sum-eccentricity energy of a graph is rational then it must be an even.

Key words and phrases. Distance in graphs, Sum-eccentricity matrix, Sum-eccentricity eigenvalues, Sum-eccentricity energy of a graph.

1. Introduction

In this paper, all graphs are assumed to be finite connected simple graphs. A graph $G= (V, E)$ is a simple graph, that is, having no loops, no multiple and directed edges. As usual, we denote $n$ to be the order and $m$ to be the size of the graph $G$. For a vertex $v \in V$, the open neighborhood of $v$ in a graph $G$, denoted $N(v)$, is the set of all vertices that are adjacent to $v$ and the closed neighborhood of $v$ is $N[v]= N(v)\cup\{v\}$. The degree of a vertex $v$ in $G$ is $d(v)= |N(v)|$. The distance $d(u,v)$ between any two vertices $u$ and $v$ in a graph $G$ is the length of the shortest
A path connecting them. The eccentricity of a vertex $v \in G$ is
\[ e(v) = \max\{d(u,v): u \in V(G)\}. \]
The radius of $G$ is $r(G) = \min\{e(v): v \in V(G)\}$ and the
diameter of $G$ is $D(G) = \max\{e(v): v \in V(G)\}$. Hence $r(G) \leq e(v) \leq D(G)$, for
every $v \in V(G)$. A vertex $v$ in a connected graph $G$ is central if
\[ e(v) = r(G), \]
while a vertex $v$ in a connected graph $G$ is peripheral vertex
if $e(v) = D(G)$. A graph $G$ is called self centered graph if $e(v) = r(G) = D(G)$.
The girth of a graph $G$ is the length of the shortest cycle contained in
the graph and denoted by $g(G)$. All the definitions and terminologies
about the graph in this paragraph available in [9].

The concept energy of a graph introduced by I. Gutman [8], in (1978). Let
$G$ be a graph with $n$ vertices and $m$ edges and let $A(G) = (a_{ij})$ be the
adjacency matrix of $G$, where
\[ a_{ij} = \begin{cases} 1, & \text{if } v_i, v_j \in E, \\ 0, & \text{otherwise}. \end{cases} \]
The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a matrix $A(G)$ assumed in a non-increasing
order, are the eigenvalues of a graph $G$[10]. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, for $t \leq n$
be the distinct eigenvalues of $G$ with multiplicities $m_1, m_2, \ldots, m_t$,
respectively, the multiset of eigenvalues of $A(G)$ is called the spectrum
of $G$ and denoted by
\[ Sp(G) = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_t \\ m_1 & m_2 & \cdots & m_t \end{bmatrix} \]
As $A$ is real symmetric with zero trace, the eigenvalues of $G$ are real
with sum equal to zero [3]. The energy $E(G)$ of a graph $G$ is defined to
be the sum of the absolute values of the eigenvalues of $G$[8], i.e.,
\[ E(G) = \sum_{i=1}^{n} |\lambda_i| \]
For more details on the mathematical aspects of the theory of graph
energy we refer to [5, 7, 10] and the references therein.

C. Adiga et. al. [2], have defined the maximum degree energy $E_M(G)$ of a
graph $G$ which depends on the maximum degree matrix $M(G)$ of $G$. Let
$G$ be a simple graph with $n$ vertices $v_1, v_2, \ldots, v_n$. Then the maximum
degree matrix $M(G) = (d_{ij})$ of a graph $G$ defined as
\[ d_{ij} = \begin{cases} \max\{d(v_i),d(v_j)\}, & \text{if } v_i v_j \in E, \\ 0, & \text{otherwise}. \end{cases} \]

As \( M(G) \) is real symmetric with zero trace, then the eigenvalues of \( G \) being real with sm equal to zero.

Ahmed M. Naji et. al. [3], have defined the concept of maximum eccentricity matrix \( M_e(G) \) of a connected graph \( G \). They obtained the maximum eccentricity energy \( EM_e(G) \) of a graph depends on the maximum eccentricity matrix. Let \( G \) be a simple connected graph with \( n \) vertices \( v_1, v_2, \ldots, v_n \) and let \( e(v_i) \) be the eccentricity of a vertex \( v_i \), \( i = 1, 2, \ldots, n \). The maximum eccentricity matrix of \( G \) defined as \( M_e(G) = (e_{ij}) \), where

\[ e_{ij} = \begin{cases} \max\{e(v_i),e(v_j)\}, & \text{if } v_i v_j \in E, \\ 0, & \text{otherwise}. \end{cases} \]

Motivated by those papers, we introduce the concept of the sum-eccentricity matrix \( S_e(G) \) of a graph \( G \) and obtain some coefficients of the characteristic polynomial \( P(G, \lambda) \) of the sum-eccentricity matrix of \( G \). We also introduce the sum-eccentricity energy \( ES_e(G) \) of a graph \( G \). Sum-eccentricity energies of some well-known graphs are obtained. Upper and lower bounds for \( ES_e(G) \) are established. It is shown that if the sum-eccentricity energy of a graph is rational then it must be an even.

2. **THE SUM-ECCENTRICITY ENERGY OF GRAPHS**

**Definition 2.1.** Let \( G \) be a graph with \( n \) vertices. Then the sum-eccentricity matrix of a graph \( G \) denoted by \( S_e(G) \), is defined as \( S_e(G) = (s_{ij}) \), where

\[ s_{ij} = \begin{cases} e(v_i) + e(v_j), & \text{if } v_i v_j \in E, \\ 0, & \text{otherwise}. \end{cases} \]

The characteristic polynomial of the sum-eccentricity matrix \( S_e(G) \) is defined by

\[ P(G, \lambda) = \det(\lambda I - S_e(G)), \]

Where \( I \) is the unt matrix of order \( n \) The eigenvalues of the sum-eccentricity matrix \( S_e(G) \) are the roots of the charakteristic polynomial of
Since $S_e(G)$ is real symmetric with zero trace, its eigenvalues must be real with sum equal to zero, i.e., $\text{trace}(S_e(G)) = 0$. We label the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in a non-increasing manner $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The sum-eccentricity energy of a graph $G$ is denoted by $ES_e(G)$ and is defined as the summation of the absolute value of the eigenvalues

$$ES_e(G) = \sum_{i=1}^{n} |\lambda_i|.$$ 

The following examples explain the concept.

**Example 2.2.** Let $G_1$ be the graph as in figure 1.

![Figure 1: $G_1$](image)

Then the sum-eccentricity matrix of $G_1$ is

$$S_e(G_1) = \begin{bmatrix} 0 & 6 & 0 & 0 & 5 & 0 \\ 6 & 0 & 5 & 0 & 5 & 0 \\ 0 & 5 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 4 & 5 \\ 5 & 5 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of $S_e(G_1)$ is

$$P(G_1, \lambda) = \lambda^6 - 168\lambda^4 - 300\lambda^3 + 4952\lambda^2 + 7500\lambda - 15625.$$ 

The sum-eccentricity eigenvalues of $G_1$ are $\lambda_1 = 12.54, \lambda_2 = 5.488, \lambda_3 = 1.2211, \lambda_4 = -2.8779, \lambda_5 = -6.6336, \lambda_6 = -9.7383$. The sum-eccentricity energy of $G_1$ is $ES_e(G_1) = 38.499$.

**Example 2.3.** Let $G_2$ be the $K_5$ graph.

Then the sum-eccentricity matrix of $G_2$ is

$$S_e(G_2) = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 2 & 0 \end{bmatrix}$$
The characteristic polynomial of \( S_e(G_2) \) is
\[
P(G_2, \lambda) = \lambda^5 - 40\lambda^3 - 16\lambda^2 - 240\lambda - 128 = (\lambda + 2)^4 (\lambda - 8).
\]
The sum-eccentricity eigenvalues of \( G_2 \) are
\[
\lambda_1 = 8, \lambda_2 = -2, \lambda_3 = -2, \lambda_4 = -2, \lambda_5 = -2.
\]
The sum-eccentricity energy of \( G_2 \) is
\[
ES_e(G_2) = 16.
\]

3. **BOUNDS FOR SUM-ECCENTRICITY ENERGY AND SUM-ECCENTRICITY EIGENVALUES**

We now give the explicit expression for the coefficient \( c_i \) of \( \lambda^{n-i} \) \((i = 0, 1, 2, 3 \text{ and } n)\) in the characteristic polynomial of the sum-eccentricity matrix \( S_e(G) \).

**Theorem 3.1.** Let \( G \) be a graph of order \( n \) and let
\[
P(G, \lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_n,
\]
be the characteristic polynomial of \( S_e(G) \). Then

1. \( c_0 = 1 \).
2. \( c_1 = 0 \).
3. \( c_2 = -\sum_{i=1,j \neq j}^{n}(e(v_i) + e(v_j))^2, \text{ where } v_i, v_j \in E. \)
4. \( c_3 = -2 \sum_{\Delta v_i v_j, 1 \leq i < j \leq n}^{n}(2e(v_i)e(v_j)e(v_k)+e(v_i)^2e(v_j)+e(v_j)^2e(v_i)+e(v_i)^2e(v_j)+e(v_j)^2e(v_i)+e(v_k)^2e(v_i)+e(v_k)^2e(v_j)). \)
5. For \( n > 1 \) we have \( c_n = (-1)^n \det(S_e(G)). \)

**Proof.** The proof of parts (1) and (2) are similar to the proof in [2].

3. Since
\[
c_2 = \sum_{1 \leq i < j \leq n} \begin{vmatrix} 0 & s_{ij} \\ s_{ji} & 0 \end{vmatrix} = \sum_{1 \leq i < j \leq n} 0 - (s_{ij}s_{ji}) = - \sum_{1 \leq i < j \leq n} s_{ij}^2
\]
and since
\[
s_{ij} = \begin{cases} e(v_i) + e(v_j), & \text{if } v_i, v_j \in E, \\ 0, & \text{otherwise}. \end{cases}
\]
Thus \( c_2 = -\sum_{i=1,j < j}^{n}(e(v_i) + e(v_j))^2, \text{ where } v_i, v_j \in E. \)
4. We have

\[ c_2 = \sum_{1 \leq i < j < k \leq n} \begin{vmatrix} s_{ik} & s_{ij} & s_{jk} \\ s_{ji} & s_{jj} & s_{jk} \\ s_{ki} & s_{kj} & s_{kk} \end{vmatrix} \]

\[ = -2 \sum_{1 \leq i < j < k \leq n} (s_j s_k s_{jk}) \]

\[ = -2 \sum_{v_i, v_j, v_k \in V} [(e(v_i) + e(v_j))(e(v_i) + e(v_j))(e(v_j) + e(v_k))] \]

\[ = -2 \sum_{v_i, v_j, v_k \in V} (2e(v_i) e(v_j) e(v_k) + e(v_j) e(v_k) e(v_i)) + e(v_j)^2 e(v_k) + e(v_j)^2 e(v_i) + e(v_k)^2 e(v_j), \]

5. We have \( c_k = (-1)^k \sum_{k=1}^{n} (all \ k \times k \ principle \ minors) \)

hence \( c_n = (-1)^n \det(S_e(G)) \).

Example 3.2. In the graph \( G_1 \) in figure 1, the coefficient \( c_2 \) of \( \lambda^2 \) in the characteristic polynomial of \( S_e(G_1) \) is equal to

\[ -\sum_{i=1, i \neq j}^{n} (e(v_i) + e(v_j))^2, \text{ where } v_i, v_j \in E \]

\[ -[(3 + 3)^2 + (3 + 2)^2 + (3 + 2)^2 + (2 + 2)^2 + (2 + 2)^2 + (2 + 2)^2] = -168 \]

Remark 3.3. a. The number of terms in \( c_2 \) in the above theorem is equal to the number of triangles in the graph.

b. If \( g(G) \neq 3 \), then \( c_3 = 0 \).

Theorem 3.4. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the sum-eccentricity eigenvalues of a graph \( G \), then

\[ \sum_{i=1}^{n} \lambda_i^2 = -2c_2. \]

Proof. We have

\[ \sum_{i=1}^{n} \lambda_i^2 = \text{trace}(S_e^2(G)) = \sum_{i=1}^{n} \sum_{k=1}^{n} s_{ik} s_{ki} = 2 \sum_{i=1}^{n} \sum_{k \neq i} s_{ik}^2 = 2 \sum_{i, k \neq i} s_{ik}^2 \]
\[ = 2 \sum_{i=1, i \neq k}^{n} (e(v_i) + e(v_k))^2, \text{ where } v_i, v_k \in E, \]

hence

\[ \sum_{i=1}^{n} \lambda_i^2 = -2c_2. \]

**Theorem 3.5.** Let \( G=K_n \), a complete graph of order \( n, n>1 \), then \( c_2 = -2n(n-1) \).

**Proof.** We have \( c_2 = -\sum_{i=1, i \neq j}^{n} (e(v_i) + e(v_j))^2, \text{ where } v_i, v_j \in E, \)

we also have in \( K_n \) each \( e(v_i) = 1 \) so

\[ c_2 = -\sum_{i=1}^{n-1} (2 + 2) i = -4 \frac{n(n-1)}{2} = -2n(n-1). \]

**Example 3.6.** In the graph \( G_2 \), the coefficient \( c_2 \) of \( \lambda^3 \) in the characteristic polynomial of \( S_e(G_2) \) is \(-2(5)(4) = -40\).

**Corollary 3.7.** For the complete graph \( K_n \), we have

\[ \sum_{i=1}^{n} \lambda_i^2 = 4n(n-1). \]

**Theorem 3.8.** If \( G \) is a graph of order \( n \), then for any sum-eccentricity eigenvalue \( \lambda_j \), we have

\[ c_2 \geq \frac{(n-2)\lambda_j^2}{2} - 2n((n-1)^2). \]

**Proof.** We have

\[ \text{trace}(S_e^2(K_n)) = 4n(n-1) \]
by Cauchy-Schwartz inequality, we have

\[ \sum_{i=1, i \neq j}^{n} \lambda_i^2 \leq (n-1) \sum_{i=1, i \neq j}^{n} \lambda_j^2 = (n-1)(4n(n-1) - \lambda_j^2) \]

so

\[ \sum_{i=1, i \neq j}^{n} \lambda_i^2 \leq 4n(n-1)^2 - \lambda_j^2(n-1) \]

i.e. \[ \sum_{i=1}^{n} \lambda_i^2 \leq 4n(n-1)^2 - \lambda_j^2(n-1) + \lambda_j^2 = 4n(n-1)^2 - \lambda_j^2(n-2). \]

Using theorem 3.4., we get

\[ c_2 \geq \frac{(n-2)\lambda_j^2}{2} - 2n(n-1)^2. \]

**Theorem 3.9.** We have

\[ \sqrt{2 \sum_{i=1, i \neq j}^{n} (e(v_i) + e(v_j))^2 + n(n-1)L^2} \leq ES_e(G) \leq \sqrt{\frac{2n^2 c_2 + 4n^3(n-1)^2}{n-2}}, \]

where \( v_i, v_j \in E, L = \prod_{i=1}^{n} \lambda_i \) and \( n > 2 \) for the left side of the inequality.

**Proof.** We have

\[ E^2S_e(G) = (\sum_{i=1}^{n} |\lambda_i|)^2 = \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i \parallel \lambda_j|. \]

Using the last inequality in theorem 3.1 and Arithmatic mean, Geometric mean inequality we get

\[ E^2S_e(G) = 2 \sum_{i=1, i \neq j}^{n} (e(v_i) + e(v_j))^2 + \sum_{i \neq j} |\lambda_i \parallel \lambda_j|, \text{ where } v_i, v_j \in E, \]

but
\[
\sum_{i<j} |\lambda_i - \lambda_j| = |\lambda_1| + |\lambda_2| + ... + |\lambda_n| \\
+ |\lambda_2| + ... + |\lambda_n| \\
+ |\lambda_n| + ... + |\lambda_n| \\
\vdots \\
+ |\lambda_n| + |\lambda_2| + ... + |\lambda_{n-1}| \\
\geq n(n-1)[|\lambda_1| + |\lambda_2| + ... + |\lambda_n|]^{1/2} (|\lambda_1| + |\lambda_2| + ... + |\lambda_n|)^{1/2} \leq n^{(n-1)} \]

\[
\sqrt{2} \sum_{i=1, i<j} (e(v_i) + e(v_j))^2 + n(n-1)L^2 \leq ES_e(G),
\]

where \(v_i, v_j \in E\) and \(L = \prod_{i=1}^{n} \lambda_i\).

On the other hand, using the previous theorem we have

\[
|\lambda_j| \leq \sqrt{\frac{2c_2 + 4n(n-1)^2}{n-2}},
\]

so

\[
\sum_{j=1}^{n} |\lambda_j| \leq \sqrt{\frac{2n^2c_2 + 4n^3(n-1)^2}{n-2}}, \text{ where } n > 2.
\]

**Theorem 3.10.** If the sum-eccentricity energy of a graph \(G\) is rational, then it must be an even integer.

**Proof.** Let \(\lambda_1, \lambda_2, ..., \lambda_n\) be the sum-eccentricity eigenvalues of a graph \(G\) with order \(n\). Then we have \(\sum_{i=1}^{n} \lambda_i = 0\). Let \(\lambda_1, \lambda_2, ..., \lambda_r\) be positive, and \(\lambda_{r+1}, \lambda_{r+2}, ..., \lambda_n\) are non-positive. Then,
\[ ES_e(G) = 2(\lambda_1 + \lambda_2 + \ldots + \lambda_r). \]

Since \( \lambda_1, \lambda_2, \ldots, \lambda_r \) are algebraic numbers, so is there sum, and hence must be integer if \( ES_e(G) \) is rational. Thus \( ES_e(G) \) is an even positive integer if it is rational.

4. THE SUM-ECCENTRICITY ENERGY FOR SOME STANDARD GRAPHS

In this section we investigate the exact values of the sum-eccentricity energy of some well-known graphs.

**Theorem 4.1.** For the cycle \( C_n, n \geq 3 \), is we have

\[
c_2 = \begin{cases} 
-n^3, & \text{if } n \text{ is even,} \\
-n(n-1)^2, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Proof.** We have \( c_2 = -\sum_{i=1, i<j}^n (e(v_i) + e(v_j))^2 \),

and

\[
e(v_i) = \begin{cases} 
n \frac{2}{2}, & \text{if } n \text{ is even,} \\
(n-1) \frac{2}{2}, & \text{if } n \text{ is odd,}
\end{cases}
\]

so if \( n \) is even \( c_2 = -\sum_{i=1, i<j}^n \left(2 \frac{n}{2}\right)^2 = -n^3 \),

and if \( n \) is odd \( c_2 = -\sum_{i=1, i<j}^n \left(2 \frac{n-1}{2}\right)^2 = -n(n-1)^2 \),

thus

\[
c_2 = \begin{cases} 
-n^3, & \text{if } n \text{ is even,} \\
-n(n-1)^2, & \text{if } n \text{ is odd.}
\end{cases}
\]
Theorem 4.1. The sum-eccentricity eigenvalues for the complete graph $K_n$ are $-2$ and $2(n-1)$ with multiplicities $(n-1)$ and $1$ respectively, and the sum-eccentricity energy for $K_n$ is $4(n-1)$.

Proof. We have

$$
|\lambda I - S_e(K_n)| = \begin{vmatrix}
\lambda & -2 & -2 & \cdots & -2 \\
-2 & \lambda & -2 & \cdots & -2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-2 & -2 & -2 & \cdots & \lambda \\
\end{vmatrix}
= (\lambda + 2)^{n-1} - 1 \begin{vmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{vmatrix}
= (\lambda + 2)^{n-1} (\lambda - 2(n-1)).
$$

The sum-eccentricity eigenvalues of $K_n$ are $\lambda_1 = 2(n-1), \lambda_2 = -2, \lambda_3 = -2, \ldots, \lambda_n = -2$, i.e., $-2$ with multiplicity $n-1$ and $2(n-1)$ with multiplicity $1$.

Hence $ES_e(K_n) = 4(n-1)$.

References


