Geometric multiaxial representation of N-qubit mixed symmetric separable states

Suma SP,[1,*] Swarnamala Sirsi,[1] Subramanya Hegde,[2] and Karthik Bharath[3]

[1]*Department of Physics, Yuvaraja's College, University of Mysore, Mysore, Karnataka 570005, India*

[2]*School of Physics, Indian Institute of Science Education and Research, Thiruvananthapuram, Thiruvananthapuram, Kerala 695016, India*

[3]*School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom*

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The study of N-qubit mixed symmetric separable states is a longstanding challenging problem as no unique separability criterion exists. In this regard, we take up the N-qubit mixed symmetric separable states for a detailed study as these states are of experimental importance and offer an elegant mathematical analysis since the dimension of the Hilbert space is reduced from 2^N to $N+1$. Since there exists a one-to-one correspondence between the spin-j system and an N-qubit symmetric state, we employ Fano statistical tensor parameters for the parametrization of the spin-density matrix. Further, we use a geometric multiaxial representation (MAR) of the density matrix to characterize the mixed symmetric separable states. Since the separability problem is NP-hard, we choose to study it in the continuum limit where mixed symmetric separable states are characterized by the P-distribution function $\lambda(\theta,\phi)$. We show that the N-qubit mixed symmetric separable states can be visualized as a uniaxial system if the distribution function is independent of θ and ϕ. We further choose a distribution function to be the most general positive function on a sphere and observe that the statistical tensor parameters characterizing the N-qubit symmetric system are the expansion coefficients of the distribution function. As an example for the discrete case, we investigate the MAR of a uniformly weighted two-qubit mixed symmetric separable state. We also observe that there exists a correspondence between the separability and classicality of states.

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I. INTRODUCTION

The study of separable states is the cornerstone of the entanglement problem. Bell inequalities were first used for the identification of entanglement. The most operationally convenient criterion for the detection of entanglement is given by Peres and Horodecki and is called the positive partial transpose (PPT) criterion [1], which is necessary and sufficient for 2×2 and 2×3 systems only [2]. There exist other criteria in the literature for detecting entanglement. Among them is the realignment criterion [3], which exhibits a powerful PPT entanglement detection capability. The entanglement witness [2,4] and uncertainty relations [5] pose operational difficulties as they depend on the expectation value of some observables for the state in question.

In practice, we deal with mixed states rather than pure states due to decoherence effects and hence it is of great importance to study mixed separable states. There exist many important papers [6–14] for mixed states in the literature; classification of local unitary equivalent classes of symmetric N-qubit mixed states and an algorithm to identify pure separable states [15] based on the geometrical multiaxial representation (MAR) of the density matrix [16] have been investigated. Makhlin [17] has presented a complete set of 18 local polynomial invariants of two-qubit mixed states and demonstrated the usefulness of these invariants in the study of entanglement. Also, detection of multipartite entanglement has been studied in depth (see, for example, [18–20]). Geometric entanglement properties of pure symmetric N-qubit states are studied in detail [21]. To this day, no generally accepted theory exists for the classification and quantification of entanglement for mixed states.

A general N-qubit mixed state resides in the Hilbert space of dimension $2^N \otimes 2^N$, which makes the mathematical computations complicated except for the lower-N values, whereas permutationally symmetric N-qubit mixed states residing in $(N+1)$-dimensional Hilbert space not only offer an elegant mathematical analysis but are also useful in a variety of quantum information tasks. They occur naturally as ground states in some Bose-Hubbard models and are the most experimentally investigated states. However, the entanglement or the separability criteria of these states have been explored relatively little and the investigation is mostly restricted to, e.g., W and Greenberger-Horne-Zeilinger states [22–24]. Recently, Bohnet-Waldraff *et al.* [25] studied the PPT separability criterion for symmetric states of multiqubit systems in terms of matrix inequalities. They established a correspondence between classical spin states and symmetric separable states. An analytical expression for the quantumness of the pure spin-1 state or equivalently the two-qubit pure symmetric state using the Majorana representation of the density matrix is given in [26]. Further, this has been extended numerically to provide an upper bound of quantumness for mixed states. The Majorana representation cannot be extended naturally to study mixed symmetric states. Therefore, in this paper we employ the little-known geometric MAR of the spin-j system to study the separability problem of mixed symmetric states. This method can also be used to investigate the quantumness of such states analytically.

This paper is organized as follows. In Sec. II we discuss the correspondence between symmetric states and spin systems. In Sec. III we explain the decomposition of the density matrix in terms of the well-known Fano statistical tensor parameters. Section IV contains the description of the multiaxial representation of pure and mixed density matrices. Section V consist of two propositions that illustrate the conditions to be satisfied

*sumarkr@gmail.com

by the mixed symmetric separable density matrix. Section VI deals with the multiaxial representation of mixed symmetric separable states and their characterization. A summary is given in Sec. VII.

II. CORRESPONDENCE BETWEEN SYMMETRIC STATES AND SPIN SYSTEMS

The N-qubit states of a set that remains unchanged by permutation of individual particles are called symmetric states, that is, $\pi_{i,j}\rho^{\text{symm}}_{1,2,\ldots,N} = \rho^{\text{symm}}_{1,2,\ldots,N}\pi_{i,j} = \rho^{\text{symm}}_{1,2,\ldots,N}$, where $\pi_{i,j}$ is called the permutation operator, with $i \neq j = 1,2,\ldots,N$. A general N-qubit state belongs to the Hilbert space $C^{2^{\otimes N}}$ and is represented by a density matrix of dimension $2^N \times 2^N$. An N-qubit symmetric state has a one-to-one correspondence with a spin-j state where $j = \frac{N}{2}$. Therefore, the $(N+1)$-dimensional symmetric subspace can be identified with a $(2j+1)$-dimensional Hilbert space that is the carrier space of the angular momentum operator $\mathbf{J}$. We focus on such symmetric states in this article as they are of considerable interest.

III. FANO REPRESENTATION OF THE SPIN-j ASSEMBLY

A general spin-j density matrix can be represented in terms of statistical tensor parameters [27–30] t^k_q:

$$\rho(\vec{J}) = \frac{\text{Tr}(\rho)}{2j+1}\sum_{k=0}^{2j}\sum_{q=-k}^{+k} t^k_q \tau^{k^\dagger}_q(\vec{J}), \tag{1}$$

where $\vec{J}$ is the angular momentum operator with components J_x, J_y, J_z. The operators τ^k_q (with $\tau^0_0 = I$ the identity operator) are irreducible tensor operators of rank k in the $(2j+1)$-dimensional angular momentum space with projection q along the axis of quantization in $\mathbb{R}^3$. The elements of τ^k_q in the angular momentum basis $|jm\rangle$, $m = -j, \ldots, +j$, are given by $\langle jm'|\tau^k_q(\vec{J})|jm\rangle = [k]C(jkj;mqm')$, where $C(jkj;mqm')$ are the Clebsch-Gordan coefficients and $[k] = \sqrt{2k+1}$. The τ^k_q satisfy the orthogonality relations

$$\text{Tr}\big(\tau^{k^\dagger}_q \tau^{k'}_{q'}\big) = (2j+1)\delta_{kk'}\delta_{qq'},$$

where $\tau^{k^\dagger}_q = (-1)^q \tau^k_q$ and

$$t^k_q = \text{Tr}\big(\rho\tau^k_q\big) = \sum_{m=-j}^{+j}\langle jm|\rho\tau^k_q|jm\rangle.$$

Since ρ is Hermitian and $\tau^{k^\dagger}_q = (-1)^q\tau^k_{-q}$, the complex conjugates t^k_q satisfy the condition $t^{k^*}_q = (-1)^q t^k_{-q}$. Furthermore, $\rho = \rho^\dagger$ and $\text{Tr}(\rho) = 1$ imply that ρ can be specified by n^2-1 independent parameters where $n = 2j+1$ is the dimension of the Hilbert space. Under rotations, the spherical tensor parameters t^k_q transform elegantly as

$$\big(t^k_q\big)^R = \sum_{q'=-k}^{+k} D^k_{q'q}(\phi,\theta,\psi)t^k_{q'},$$

where $D^k_{q'q}(\phi,\theta,\psi)$ is the (q',q) element of the Wigner D matrix and (ϕ,θ,ψ) are the Euler angles.

IV. MULTIAXIAL REPRESENTATION OF PURE AND MIXED STATES

The spherical tensor parameters t^k_q of a spin-j state possess a geometric representation called the multiaxial representation [16], which is similar to the Majorana representation. The Majorana representation is applicable to pure symmetric states only, whereas the MAR is applicable for general mixed spin-j states as well as pure states. The MAR is characterized by the Euler angles (θ,ϕ,ψ), which are related to the parameters t^k_q in the following manner. Consider a rotation $R(\phi,\theta,0)$ of the frame of reference such that t^k_k in the rotated frame vanishes:

$$\big(t^k_k\big)^R = 0 = \sum_{q=-k}^{+k} D^k_{qk}(\phi,\theta,0)t^k_q.$$

This implies that by using the Wigner expression for D^j matrices [31], we obtain the polynomial equation

$$\begin{aligned}\chi(\theta,\phi) &= \sum_{q=-k}^{k} e^{-iq\phi}(-1)^{k-q}\sqrt{\binom{2k}{k+q}}t^k_q \\ &\quad\times\left(\cos\frac{\theta}{2}\right)^{k+q}(-1)^{k-q}\left(\sin\frac{\theta}{2}\right)^{k-q} \\ &= \mathcal{A}\sum_{q=-k}^{+k}\sqrt{\binom{2k}{k+q}}t^k_q Z^{k-q} = 0,\end{aligned} \tag{2}$$

where $Z = \tan(\frac{\theta}{2})e^{i\phi}$ and the overall coefficient

$$\mathcal{A} = \cos^{2k}\left(\frac{\theta}{2}\right)e^{-ik\phi}.$$

A trivial solution is $\theta = \pi$. We therefore redefine $\chi(\cdot,\cdot)$ suitably as a polynomial in Z as

$$P_1(Z) = \sum_{q=-k}^{+k}\sqrt{\binom{2k}{k+q}}t^k_q Z^{k-q} = 0. \tag{3}$$

Alternatively, it is possible to redefine $\chi(\cdot,\cdot)$ as a polynomial P_2 in $Z' = \frac{1}{Z} = \cot(\frac{\theta}{2})e^{-i\phi}$, with

$$P_2(Z') = \sum_{q=-k}^{+k}\sqrt{\binom{2k}{k+q}}t^k_q Z'^{k+q} = 0, \tag{4}$$

by ignoring the trivial solution $\theta = 0$. In both cases, every k leads to $2k$ solutions,

$$\{(\theta_1,\phi_1),\ldots,(\theta_k,\phi_k),(\pi-\theta_1,\pi+\phi_1),\ldots,(\pi-\theta_k,\pi+\phi_k)\}.$$

Thus the $2k$ solutions constitute k axes or k double-headed arrows: For every solution (θ_i,ϕ_i), $(\pi-\theta_i,\pi+\phi_i)$ also forms a solution. The solution set of P_1 (equivalently P_2) provides the key insight into the geometrical interpretation of the spherical tensor parameters t^k_q, elucidated as follows. For a fixed (θ_i,ϕ_i), $i = 1,\ldots,k$, consider a unit vector $\hat{Q}_i := \hat{Q}(\theta_i,\phi_i)$ in $\mathbb{R}^3$. Define

$$s^k_q = (\{\cdots[(\hat{Q}_1\otimes\hat{Q}_2)^2\otimes\hat{Q}_3]^3\otimes\cdots\otimes\hat{Q}_{k-1}\}^{k-1}\otimes\hat{Q}_k)^k_q,$$

where

$$(\hat{Q}_1 \otimes \hat{Q}_2)^2_q = \sum_{q_1} C(11k; q_1 q_2 q)(\hat{Q}_1)^1_{q_1} (\hat{Q}_2)^1_{q_2}$$

and the spherical components of $\hat{Q}$ are given by

$$[\hat{Q}(\theta,\phi)]^1_q = \sqrt{\frac{4\pi}{3}} Y^1_q(\theta,\phi).$$

Here $Y^1_q(\theta,\phi)$ are the well-known spherical harmonics.

As a consequence, we can state that

$$t^k_q = r_k(\{\cdots[(\hat{Q}_1 \otimes \hat{Q}_2)^2 \otimes \hat{Q}_3]^3 \otimes \cdots \otimes \hat{Q}_{k-1}\}^{k-1} \otimes \hat{Q}_k)^k_q.$$

Thus, in the MAR, the symmetric state of the N-qubit assembly can be represented geometrically by a set of $N = 2j$ spheres of radii $r_1, r_2, \ldots, r_k$ corresponding to each value of k. The kth sphere in general consists of a constellation of $2k$ points on its surface specified by $\hat{Q}_i := \hat{Q}(\theta_i, \phi_i)$ and $\hat{Q}(\pi - \theta_i, \pi + \phi_i)$, $i = 1, 2, \ldots, k$. In other words, for a fixed k, every t^k_q, $q = -k, -k+1, \ldots, 0, 1, \ldots, k$, is specified by k axes in a sphere of radius r_k.

V. MIXED SYMMETRIC SEPARABLE STATES

Before employing the MAR to develop a criterion for separability, we examine some properties of mixed symmetric separable states. By definition, an N-qubit state is said to be fully separable if it can be decomposed as $\rho = \sum_{i=1}^n \lambda_i \rho^1_i \otimes \rho^2_i \otimes \cdots \otimes \rho^N_i$, where for ρ^α_i, $\alpha = 1, \ldots, N$ is the ith decomposition of the system with α as the qubit index. The following propositions elucidate the relationships between separable, mixed, and symmetric states.

Proposition 1. Consider $0 \leqslant \lambda_i \leqslant 1$, $i = 1, \ldots, n$, with $\sum_{i=1}^n \lambda_i = 1$. An N-qubit fully separable state $\rho = \sum_{i=1}^n \lambda_i \rho^1_i \otimes \rho^2_i \otimes \cdots \otimes \rho^N_i$, where $\rho^k_i, k = 1, \ldots, N$ is the density matrix of the pure state for the kth particle, is mixed if $n \geqslant 2$.

Proof. Consider the density matrix for the αth qubit where $\alpha = 1, \ldots, N$, defined as $\rho^\alpha_i = \frac{1}{2}[I + \vec{\sigma} \cdot \vec{p}_i(\alpha)]$, $i = 1, 2, \ldots, n$, where $\vec{p}_i(\alpha)$ is the polarization vector characterizing the αth qubit in the ith decomposition. For ρ to be pure, $\mathrm{Tr}\rho^2 = 1$, which implies that

$$\sum_{i,j} \lambda_i \lambda_j \mathrm{Tr}(\rho^1_i \rho^1_j) \mathrm{Tr}(\rho^2_i \rho^2_j) \cdots \mathrm{Tr}(\rho^N_i \rho^N_j) = 1.$$

Therefore,

$$\sum_{i,j} \lambda_i \lambda_j [1 - \mathrm{Tr}(\rho^1_i \rho^1_j) \mathrm{Tr}(\rho^2_i \rho^2_j) \cdots \mathrm{Tr}(\rho^N_i \rho^N_j)] = 0.$$

Consequently $\mathrm{Tr}(\rho^\alpha_i \rho^\alpha_j) = \frac{1}{2}[I + \vec{p}_i(\alpha) \cdot \vec{p}_j(\alpha)] < 1$ since $\vec{p}_i(\alpha) \cdot \vec{p}_j(\alpha) < 1$ for $\alpha = 1, 2, \ldots, N$, which implies that

$$\sum_{i,j} \lambda_i \lambda_j [1 - \mathrm{Tr}(\rho^1_i \rho^1_j) \mathrm{Tr}(\rho^2_i \rho^2_j) \cdots \mathrm{Tr}(\rho^N_i \rho^N_j)] > 0,$$

owing to

$$1 - \mathrm{Tr}(\rho^1_i \rho^1_j) \mathrm{Tr}(\rho^2_i \rho^2_j) \cdots \mathrm{Tr}(\rho^N_i \rho^N_j) > 0, \quad \lambda_i > 0.$$

Therefore,

$$\sum_{i,j} \lambda_i \lambda_j [1 - \mathrm{Tr}(\rho^1_i \rho^1_j) \mathrm{Tr}(\rho^2_i \rho^2_j) \cdots \mathrm{Tr}(\rho^N_i \rho^N_j)] \neq 0,$$

implying that ρ cannot be pure. However, ρ can be pure if $\vec{p}_i(\alpha) \cdot \vec{p}_j(\alpha) = 1$ for all i, j, in which case there is only one term

$$\rho = \rho^1 \otimes \rho^2 \otimes \cdots \otimes \rho^N.$$

Therefore, we may define a separable mixed state as

$$\rho = \sum_{i=1}^n \lambda_i \rho^1_i \otimes \rho^2_i \otimes \cdots \otimes \rho^N_i,$$

where $n > 1$ and $\rho^1_i, \rho^2_i, \ldots, \rho^N_i$ are pure. ■

Proposition 2. An N-qubit fully separable mixed state

$$\rho = \sum_{i=1}^n \lambda_i \rho^1_i \otimes \rho^2_i \otimes \cdots \otimes \rho^N_i \tag{5}$$

is permutationally symmetric if $\rho^1_i = \rho^2_i = \cdots = \rho^N_i$.

Proof. Now let us see if symmetrization of two different states ρ^1_i and ρ^2_i leads to a state in symmetric subspace. For some fixed λ_i in (5), consider the first two terms $\rho^1_i \otimes \rho^2_i$ and define

$$\rho^{12}_i := \frac{\rho^1_i \otimes \rho^2_i + \rho^2_i \otimes \rho^1_i}{2}.$$

Evidently, ρ^{12}_i is a density matrix. Let $\rho^1_i = \frac{I + \vec{\sigma} \cdot \vec{p}_i(1)}{2}$ and $\rho^2_i = \frac{I + \vec{\sigma} \cdot \vec{p}_i(2)}{2}$. For notational convenience we set $\vec{p}_i(\alpha) = \vec{p}^\alpha_i$, $p^2_{i-} = p^2_{ix} - i p^2_{iy}$, $p^1_{i-} = p^1_{ix} - i p^1_{iy}$, $p^2_{i+} = p^2_{ix} + i p^2_{iy}$, and $p^1_{i+} = p^1_{ix} + i p^1_{iy}$. Then

$$\frac{\rho^1_i \otimes \rho^2_i + \rho^2_i \otimes \rho^1_i}{2}$$
$$= \begin{bmatrix} \frac{(1+p^2_{iz})(1+p^1_{iz})}{4} & \frac{p^2_{i-}(1+p^1_{iz})+p^1_{i-}(1+p^2_{iz})}{8} & \frac{p^1_{i-}(1+p^2_{iz})+p^2_{i-}(1+p^1_{iz})}{8} & \frac{p^1_{i-}p^2_{i-}}{4} \\ \frac{p^2_{i+}(1+p^1_{iz})+p^1_{i+}(1+p^2_{iz})}{8} & \frac{(1-p^2_{iz})(1+p^1_{iz})+(1-p^1_{iz})(1+p^2_{iz})}{8} & \frac{p^1_{i-}p^2_{i+}+p^2_{i-}p^1_{i+}}{8} & \frac{p^1_{i-}(1-p^2_{iz})+p^2_{i-}(1-p^1_{iz})}{8} \\ \frac{p^1_{i+}(1+p^2_{iz})+p^2_{i+}(1+p^1_{iz})}{8} & \frac{p^1_{i+}p^2_{i-}+p^2_{i+}p^1_{i-}}{8} & \frac{(1-p^1_{iz})(1+p^2_{iz})+(1-p^2_{iz})(1+p^1_{iz})}{8} & \frac{p^2_{i-}(1-p^1_{iz})+p^1_{i-}(1-p^2_{iZ})}{8} \\ \frac{p^2_{i+}p^1_{i+}}{4} & \frac{p^1_{i+}(1-p^2_{iz})+p^2_{i+}(1-p^1_{iz})}{8} & \frac{(1-p^1_{iz})p^2_{i+}+(1-p^2_{iz})p^1_{i+}}{8} & \frac{(1-p^1_{iz})(1-p^2_{iz})}{4} \end{bmatrix}$$

in the computational bases $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$. We can choose a set of bases called angular momentum bases given by $\{|11\rangle = |\uparrow\uparrow\rangle, |10\rangle = \frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}, |1-1\rangle = |\downarrow\downarrow\rangle, |00\rangle = \frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}\}$, out of which the first three basis states are permutationally symmetric and the last one is permutationally antisymmetric. The unitary transformation that connects the computational basis set to the above set is

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}.$$

The elements of unitary transformation are the Clebsch-Gordan (CG) coefficients. It is very well known in angular momentum theory that the Clebsch-Gordan addition of two angular momenta j_1 and j_2 resulting in the angular momentum j is given by

$$|j_1 j_2 j m\rangle = \sum_{m_1,m_2} C(j_1 j_2 j; m_1 m_2 m)|j_1 m_1\rangle|j_2 m_2\rangle.$$

Thus, the two-qubit symmetric state has a one-to-one correspondence with a spin-1 state and the most general two-qubit state resides in $2^2 = 4$ dimensional Hilbert space. Thus, the Clebsch-Gordan decomposition of the space is given by $2 \otimes 2 = 3 \oplus 1$, where the highest, that is, the three-dimensional, space is the symmetric subspace. Now transforming the density matrix to symmetric subspace, we get

$$U\rho U^\dagger = \begin{bmatrix} \frac{(1+p^2_{iz})(1+p^1_{iz})}{4} & \frac{p^1_{i-}(1+p^1_{iz})+p^1_{i-}(1+p^2_{iz})}{4\sqrt{2}} & \frac{p^1_{i-}p^2_{i-}}{4} & 0 \\ \frac{p^2_{i+}(1+p^1_{iz})+p^1_{i+}(1+p^2_{iz})}{4\sqrt{2}} & \frac{(1-p^2_{iz}p^1_{iz}+p^1_{ix}p^2_{ix}+p^1_{iy}p^2_{iy})}{4} & \frac{p^1_{i-}(1-p^2_{iz})+p^2_{i-}(1-p^1_{iz})}{4\sqrt{2}} & 0 \\ \frac{p^1_{i+}p^2_{i+}}{4} & \frac{p^2_{i+}(1-p^1_{iz})+p^1_{i+}(1-p^2_{iz})}{4\sqrt{2}} & \frac{(1-p^2_{iz})(1-p^1_{iz})}{4} & 0 \\ 0 & 0 & 0 & \frac{1-\hat{p}^2_i\hat{p}^1_i}{4} \end{bmatrix}.$$

Thus symmetrization of ρ^1_i and ρ^2_i does not lead to a state in symmetric subspace unless $\frac{1-\hat{p}^2_i\hat{p}^1_i}{4} = 0$. This implies that $\hat{p}^2_i = \hat{p}^1_i$. Similarly, by continuing in the same way, considering the permutational symmetry of all N qubits taking two qubits at a time, we get $\hat{p}^1_i = \hat{p}^2_i = \hat{p}^3_i = \cdots = \hat{p}^N_i$. Thus all the N qubits in a partition will have the same vector polarization. Therefore, a mixed symmetric separable state is an ensemble of symmetric pure separable states and henceforth we write it as $\rho = \sum_i \lambda_i \rho_i \otimes \rho_i \otimes \cdots \otimes \rho_i$. ■

VI. MULTIAXIAL REPRESENTATION OF MIXED SYMMETRIC SEPARABLE STATES

To arrive at the multiaxial representation of an N-qubit symmetric separable state, let us consider

$$\rho = \sum_i^n \lambda_i \rho_i \otimes \rho_i \otimes \cdots \otimes \rho_i = \sum_i^n \lambda_i \varrho_i^N,$$

where $\varrho_i^N = \rho_i \otimes \rho_i \otimes \ldots \otimes \rho_i$. The unitary transformation $\mathcal{U}$ decomposes ϱ_i^N into the direct sum of its composite density matrices, out of which the $(2j+1)$- or the $[2(\frac{N}{2})+1]$-dimensional density matrix ρ_i' is totally symmetric and the rest are zeros. The elements of $\mathcal{U}$ are the well-known CG coefficients. Therefore, ρ in symmetric subspace is written as

$$\rho^j_{\text{symm}} = \sum_i \lambda_i \rho_i'. \tag{6}$$

From Eq. (1),

$$\rho^j_{\text{symm}} = \frac{1}{2j+1}\sum_{kq} t^k_q \tau^{k^\dagger}_q = \frac{1}{2j+1}\sum_i \sum_{kq} \lambda_i t^k_q(i) \tau^{k^\dagger}_q,$$

which implies that

$$t^k_q = \sum_i^n \lambda_i t^k_q(i). \tag{7}$$

Clearly, each of the ρ_i' is a pure spin-j density matrix expressed in the $|jm\rangle$ basis and the corresponding density matrix is $(\rho_i \otimes \rho_i \otimes \cdots \otimes \rho_i)$ in the computational basis, which can also be written as $|\psi_i\psi_i\cdots\psi_i\rangle\langle\psi_i\psi_i\cdots\psi_i|$. The MAR of pure symmetric separable states has already been investigated [15], which we introduce here briefly. As ρ_i' is characterized by (θ_i,ϕ_i), in a rotated frame of reference, whose z axis is parallel to (θ_i,ϕ_i), ρ_i' assumes a canonical form given by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \tag{8}$$

in the angular momentum basis. The only nonzero spherical tensor parameters characterizing the above state are

$$(t^k_0)^{\text{rotated}} = [k]\rho_{jj}C(jkj;j0j), \quad k = 0,1,2,\ldots,2j,$$

and thus each $t^k_q(i)$ is constructed out of a single axis and the resultant ρ_i' is characterized by one axis, namely, $\hat{Q}(\theta_i,\phi_i)$, and represents a uniaxial system; alternatively, $t^k_q(i) \propto Y^k_q(\theta_i\phi_i)$. Hence we write (7) as

$$t^k_q = C\sum_i \lambda_i Y^k_q(\theta_i\phi_i),$$

where C is a proportionality constant. In the continuum limit it is natural to take t^k_q as

$$t^k_q = C\int \lambda(\theta,\phi)Y^k_q(\theta,\phi)d\Omega, \tag{9}$$

where $\int \lambda(\theta,\phi)d\Omega = 1$, $\lambda(\theta,\phi)$ is positive, and $d\Omega = \sin\theta\, d\theta\, d\phi$.

Now it is interesting to investigate the functional form of t_q^k in the continuum limit. To do this, let us consider the angular momentum operators L_z and $L^2 = \vec{L}\cdot\vec{L}$, which have the form

$$L_z = -i\hbar\frac{\partial}{\partial\phi},$$
$$L^2 = -\hbar^2\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \hbar^2\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

It can be easily seen that t_q^k is a simultaneous eigenstate of L^2 and L_z if λ is independent of θ and ϕ. In such a case, for every k, the t_q^k of the mixed symmetric separable state are characterized by k axes that are collinear.

Classicality and separability

It is well known that a density matrix ρ is called P representable if it can be written as a convex sum of coherent states $\rho = \int d\alpha\, P(\alpha)\,|\alpha\rangle\,\langle\alpha|$, where α is a coherent state and $P(\alpha)$ is a probability density function with $\int P(\alpha)d\alpha = 1$. We can identify our pure separable symmetric state ρ_i' of (6) with the coherent states as coherent states are the rotated $|jj\rangle$ states [32], i.e.,

$$\begin{aligned}|\alpha(\theta,\phi)\rangle &= \sum_m |jm\rangle\,\langle jm|\,R(\phi,\theta,0)\,|jj\rangle = \sum_m D^j_{mj}(\phi,\theta,0)\,|jm\rangle\\ &= \sum_{m=-j}^{j}\sqrt{\binom{2j}{j+m}}(\sin\theta)^{j-m}(\cos\theta)^{j+m}e^{-i(j+m)\phi}\,|jm\rangle,\end{aligned}$$

where $D^j_{mj}(\phi,\theta,0)$ are Wigner D matrices.

Rotated $|jj\rangle$ states assume a canonical form as shown in Eq. (8) and hence they are pure separable states. Thus, N-qubit mixed symmetric separable states are identified with P-representable states. Any density matrix that is P representable is widely accepted as a classical state [33,34]. Hence N-qubit symmetric separable states are classical spin-$\frac{N}{2}$ states. Conversely, a classical spin-j state with $j = \frac{N}{2}$ that is a convex mixture of coherent states can be realized as an N-qubit symmetric separable state in $2j$ tensor product space, proof of which is given in [25].

It has already been proved that $P(\alpha)$ is not uniquely determined by the density operator [35]. We choose the most general positive function on the sphere $\lambda(\theta,\phi)$ as the P function [36] and study the MAR of the corresponding density matrix. If $\lambda(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a^l_m Y^{l*}_m(\theta,\phi)$ then

$$\begin{aligned}t_q^k &= \int \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a^l_m Y^{l*}_m(\theta,\phi)Y^k_q(\theta,\phi)d\Omega\\ &= \sum_{lm} a^l_m\delta_{kl}\delta_{qm} = a^k_q.\end{aligned}$$

Thus, the t_q^k are the expansion coefficients of the probability density function $\lambda(\theta,\phi)$. Given the a^k_q, we can explicitly determine the axes from the MAR.

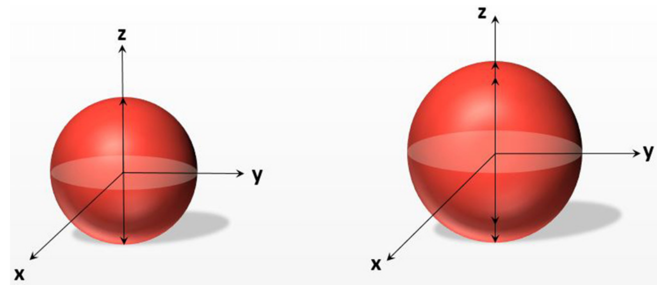

FIG. 1. Multiaxial representation of mixed symmetric separable states showing the axes characterized by t_0^k for $k = 1,2$.

Now as an example let us choose a probability density function of the form $Y^l_m(\theta,\phi)Y^{l\,*}_m(\theta,\phi)$ and study the MAR of the t_q^k belonging to the N-qubit mixed symmetric separable state, i.e.,

$$t_q^k = C\int Y^l_m(\theta,\phi)Y^{l\,*}_m(\theta,\phi)Y^k_q(\theta,\phi)d\Omega.$$

Using Eq. (11) of Sec. (5.6) of [31],

$$\begin{aligned}t_q^k = C\int\sum_{LL'}(-1)^m&\sqrt{\frac{(2l+1)^2(2k+1)}{(4\pi)^2(2L+1)}}\\ &\times C(llL':000)C(L'kL:000)C(llL':m-m0)\\ &\times C(L'kL:0qq)Y^L_q(\theta,\phi)d\Omega\end{aligned}$$

and after integration,

$$\begin{aligned}t_q^k = C\sum_{LL'}(-1)^m&\sqrt{\frac{(2l+1)(2l+1)(2k+1)}{(4\pi)^2(2L+1)}}C(llL':000)\\ &\times C(L'kL:000)C(llL':m-m0)\\ &\times C(L'kL:0qq)\delta_{L0}\delta_{q0}\sqrt{4\pi}.\end{aligned}$$

Therefore, the only nonzero t_q^k are given by

$$\begin{aligned}t_0^k = C\sum_{L'}(-1)^m&\sqrt{\frac{(2l+1)(2l+1)(2k+1)}{(4\pi)^2(2L+1)}}C(llL':000)\\ &\times C(L'k0:000)C(llL':m-m0)\\ &\times C(L'k0:0qq)\sqrt{4\pi}.\end{aligned}$$

Thus, the N-qubit mixed separable symmetric state ρ characterized by the above t_0^k is a uniaxial system with the axes being collinear to the z axis as explained in Sec. VI (see Fig. 1).

Now let us give an example of a discrete case; a two-qubit mixed symmetric separable state with a uniform distribution is

$$\begin{aligned}\rho &= \tfrac{1}{4}(\rho_x\otimes\rho_x) + \tfrac{1}{4}(\rho_{-x}\otimes\rho_{-x})\\ &\quad+ \tfrac{1}{4}(\rho_z\otimes\rho_z) + \tfrac{1}{4}(\rho_{-z}\otimes\rho_{-z}),\end{aligned}$$

where x, $-x$, z, and $-z$ are four maximally separated points on a Bloch sphere and $\rho_i = \frac{I+\sum_i\sigma_i p_i}{2}$, $i = x,y,z$. Explicitly,

$$\rho = \frac{1}{16}\begin{bmatrix}6 & 0 & 0 & 2\\ 0 & 2 & 2 & 0\\ 0 & 2 & 2 & 0\\ 2 & 0 & 0 & 6\end{bmatrix}$$

and transforming ρ to the $|jm\rangle$ basis, we have

$$\rho^{jm} = \frac{1}{16}\begin{bmatrix} 6 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix}.$$

The nonzero t_q^k are $t_0^2 = \frac{1}{4\sqrt{2}}$, and $t_2^2 = t_{-2}^2 = \frac{\sqrt{3}}{8}$. The polynomial equation (3) for $k = 2$ becomes

$$z^4\frac{\sqrt{3}}{8} + z^2\frac{\sqrt{6}}{4\sqrt{2}} + \frac{\sqrt{3}}{8} = 0,$$

solutions of which give us the two collinear axes, namely, $\hat{Q}(\frac{\pi}{2},\frac{\pi}{2})$ and $\hat{Q}(\frac{\pi}{2},\frac{\pi}{2})$.

Therefore, in each of the above cases, $t_q^k \propto Y_q^k(\theta,\phi)$. Now that we have given a MAR for mixed symmetric separable states or equivalently for classical states, one can explore the quantumness of such states analytically.

VII. CONCLUSION

We have identified the mixed symmetric fully separable N-qubit state with a spin-j density matrix and expressed it in terms of Fano statistical tensor parameters. Using the multiaxial representation of the density matrix, we realized that a fully separable N-qubit symmetric state is characterized by spherical tensor parameters t_q^k, which are always proportional to the spherical harmonics $Y_q^k(\theta,\phi)$ in the continuum limit when the P distribution function $\lambda(\theta,\phi)$ is independent of θ and ϕ. Further, it was shown that for such a case the mixed symmetric separable states are characterized by collinear axes. In contrast, for a general density matrix each t_q^k is characterized by k distinct axes. We have also identified N-qubit mixed symmetric separable states with P-representable states or classical states. Since the distribution function is not uniquely determined by the density matrix, we have chosen it to be the most general positive function on a sphere of unit radius and concluded that the t_q^k are given by expansion coefficients of the P function. By choosing $Y_m^l Y_m^{l^*}$ as the probability density function we have proved that the corresponding state is characterized by nonzero t_0^k only. In other words, the axes are collinear. We have also examined the MAR of a two-qubit mixed symmetric state consisting of four terms with equal weight and concluded that it is characterized by collinear axes.

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