Ukrainian Mathematical Journal, Vol. 62, No. 12, May, 2011 (Ukrainian original Vol. 62, No. 12, December, 2010)

# ON A CONTINUED FRACTION OF ORDER 12

**K. R. Vasuki, Abdulrawf A. A. Kahtan, G. Sharath, and C. Sathish Kumar** UDC 511.72

We present some new relations between a continued fraction $U(q)$ of order 12 (established by M. S. M. Naika et al.) and $U(q^n)$ for $n = 7, 9, 11, 13$.

## 1. Introduction

Throughout the paper, we assume that $|q| < 1$. For a positive integer $n$, we use the following standard notation:

$$(a)_0 := (a;q)_0 = 1,$$

$$(a)_n := (a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i),$$

and

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

In Chap. 16 of his Second Notebook [12, p. 197; 3, p. 34], S. Ramanujan developed the theory of theta functions and his theta function is defined as follows:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab| < 1.$$

Following Ramanujan [12, p. 197], we define

$$\phi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{\frac{k(k+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

University of Mysore, Mysore, India.

Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 62, No. 12, pp. 1609–1619, December, 2010. Original article submitted October 27, 2009; revision submitted July 8, 2010.

 0041–5995/11/6212–1866 

and

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}} = (q;q)_\infty.$$

For the sake of convenience, we denote

$$f(-q^n) = f_n.$$

The celebrated Rogers–Ramanujan continued fraction is defined as

$$R(q) := \frac{q^{1/5} f(-q,-q^4)}{f(-q^2,-q^3)} = \frac{q^{1/5}}{1} \; \underset{+}{} \; \frac{q}{1} \; \underset{+}{} \; \frac{q^2}{1} \; \underset{+}{} \; \frac{q^3}{1} \; \underset{+}{} \; \ldots. \tag{1.1}$$

On p. 365 of his Lost Notebook [13], Ramanujan recorded five identities showing the relationships between $R(q)$ and five continued fractions $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$, and $R(q^5)$. He also recorded these identities at the scattered places of his Notebooks [12]. L. J. Rogers [14] established the modular equations relating $R(q)$ and $R(q^n)$ for $n = 2, 3, 5,$ and 11. The last of these equations cannot be found in Ramanujan's works. For the proof of these equations, we can refer the reader, e.g., to [4]. Recently K. R. Vasuki and S. R. Swamy [19] found the modular equation relating $R(q)$ with $R(q^7)$.

The Ramanujan's cubic continued fraction $G(q)$ is defined as follows:

$$G(q) := \frac{q^{1/3} f(-q,-q^5)}{f(-q^3,-q^3)} = \frac{q^{1/3}}{1} \; \underset{+}{} \; \frac{q+q^2}{1} \; \underset{+}{} \; \frac{q^2+q^4}{1} \; \underset{+}{} \; \frac{q^3+q^6}{1} \; \underset{+}{} \; \ldots. \tag{1.2}$$

The continued fraction (1.2) was first introduced by Ramanujan in his second letter to G. H. Hardy [10]. He also recorded the continued fraction (1.2) on p. 365 of his Lost Notebook [13] and claimed that there are many results for $G(q)$ similar to the results obtained for the famous Rogers–Ramanujan continued fraction (1.1).

Motivated by Ramanujan's claim, H. H. Chan [5], N. D. Baruah [1], K. R. Vasuki, and B. R. Srivatsa Kumar [18] established the modular relations between $G(q)$ and $G(q^n)$ for $n = 2, 3, 5, 7, 11$ and 13.

The Ramanujan Göllnitz–Gordon continued fraction [13, p. 44; 8; 9] is defined as follows:

$$H(q) := \frac{q^{1/2} f(-q^3,-q^5)}{f(-q,-q^7)} = \frac{q^{1/2}}{1} \; \underset{+}{} \; \frac{q^2}{1+q^3} \; \underset{+}{} \; \frac{q^4}{1+q^5} \; \underset{+}{} \; \ldots. \tag{1.3}$$

Chan and S. S. Hang [6] and K. R. Vasuki and B. R. Srivatsa Kumar [17] established the relationships between $H(q)$ and $H(q^n)$ with $n = 3, 4, 5, 11$ by using the modular equations deduced by Ramanujan. Recently, B. Cho, J. K. Koo, and Y. K. Park [7] extended the results cited above for the continued fraction (1.3) to all odd prime $p$ by computing the affine models of modular curves $X(\Gamma)$ with $\Gamma = \Gamma_1(8) \cap \Gamma_0(16p)$.

Motivated by the cited works on the continued fractions (1.1)–(1.3), in the present paper, we establish the relationship between the following continued fraction $U(q)$ and $U(q^n)$ for $n = 2, 7, 9, 11$ and 13 :

$$U(q) := \frac{q f(-q,-q^{11})}{f(-q^5,-q^7)} \frac{q(1-q)}{(1-q^3)} \; \underset{+}{} \; \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)} \; \underset{+}{} \; \frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})} \; \underset{+}{} \; \ldots. \tag{1.4}$$

The continued fraction (1.4) was established by M. S. M. Naika, et al. [11] as a special case of a fascinating continued-fraction identity recorded by Ramanujan in his Second Notebook [12, p. 74]. Furthermore, they have also established a modular relationship between the continued fraction $U(q)$ and $U(q^n)$ with $n = 3$ and 5.

## 2. Some Preliminary Results

**Theorem 2.1** [11]. *We have*

$$\frac{\phi(q)}{\phi(q^3)} = \frac{1+U(q)}{1-U(q)}. \tag{2.1}$$

**Theorem 2.2.** *We have*

$$\frac{\phi^2(-q)}{\phi^2(-q^3)} = 1 - \frac{4U(q)}{1+U^2(q)}. \tag{2.2}$$

***Proof.*** It follows from [2] that

$$\left(\phi^2(q)+\phi^2(q^3)\right)^2 = \frac{4\phi(q^3)\phi^3(-q^3)\phi(q)}{\phi(-q)}. \tag{2.3}$$

Moreover, it follows from [16] that

$$\left(3\phi^2(q^3)-\phi^2(q)\right)^2 = \frac{4\phi(q^3)\phi(q)\phi^3(-q)}{\phi(-q^3)}. \tag{2.4}$$

From (2.3) and (2.4), we deduce the relation

$$\frac{\phi^2(-q)}{\phi^2(-q^3)} = \frac{3-\dfrac{\phi^2(q)}{\phi(q^3)}}{1+\dfrac{\phi^2(q)}{\phi^2(q^3)}}.$$

By using (2.1) on the right-hand side of this identity, we find (2.2).

**Theorem 2.3.** *We have*

$$\frac{\psi^2(q^2)}{q\psi^2(q^6)} = U(q) + \frac{1}{U(q)} - 1. \tag{2.5}$$

***Proof.*** From [2], we conclude that

$$1 - \frac{\phi^2(-q)}{\phi^2(-q^3)} = \frac{4qf_1f_{12}^3}{f_4f_3^3} \tag{2.6}$$

and, in addition, that

$$1 + \frac{\psi^2(q^2)}{q\psi^2(q^6)} = \frac{f_3^3f_4}{qf_1f_{12}^3}.$$

By using (2.6) and the identity presented above, we find

$$1+\frac{\psi^2(q^2)}{q\psi^2(q^6)}=\frac{4}{1-\dfrac{\phi^2(-q)}{\phi^2(-q^3)}}.$$

In view of (2.2), this gives the required result.

**Theorem 2.4.** *We have*

$$\frac{\psi^2(q)}{q^{1/2}\psi^2(q^3)}=\frac{1+U(q)}{1-U(q)}\sqrt{U(q)+\frac{1}{U(q)}-1}. \tag{2.7}$$

***Proof.*** Consider the equality

$$\frac{\psi^4(q)}{q\psi^4(q^3)}=\frac{\phi^2(q)\psi^2(q^2)}{q\phi^2(q^3)\psi^2(q^6)}, \tag{2.8}$$

where we have used the identity $\phi(q)\psi(q^2)=\psi^2(q)$ from Entry 25 [3, p. 40]. Further, by using (2.1) and (2.7) on the right-hand side of this equality, we obtain the required result.

**Theorem 2.5.** *We have*

$$\frac{\psi^4(-q)}{q\psi^4(-q^3)}=\left[\frac{U^2(q)+1}{U(q)}+\frac{4U(q)}{1+U^2(q)}-5\right]. \tag{2.9}$$

***Proof.*** Changing $q$ into $-q$ in relation (2.8) and then using (2.2) and (2.5) on the right-hand side of the same relation, we obtain (2.9).

To prove our main results, we require the following easily deducible identities:

$$\phi(q)=\frac{f_2^5}{f_1^2f_4^2},\qquad \psi(q)=\frac{f_2^2}{f_1},$$

$$\phi(-q)=\frac{f_1^2}{f_2},\qquad \psi(-q)=\frac{f_1f_4}{f_2},$$

$$f(q)=\frac{f_2^3}{f_1f_4},\qquad \chi(q)=\frac{f_2^2}{f_1f_4},$$

and

$$\chi(-q)=\frac{f_1}{f_2}. \tag{2.10}$$

Moreover, we need the following identities analogous to the Rogers–Ramanujan forty identities:

**Theorem 2.6** [15]**.** *Let*

$$M(q)=\frac{f(-q^5,-q^7)}{f_4}$$

*and*

$$N(q)=\frac{f(-q,-q^{11})}{f_4}.$$

*Then*

$$M(q^2)N(q)+qM(q)N(q^2)=\frac{f_3f_{24}}{f_4f_8}, \tag{2.11}$$

$$M(q^2)N(q)-qM(q)N(q^2)=\frac{f_1^2f_6f_{24}}{f_2f_3f_4f_8}, \tag{2.12}$$

$$M(q^3)N(q)+q^2M(q)N(q^3)=\frac{f_1f_6^5f_9f_{36}}{f_2^2f_3^2f_{12}^3f_{18}}, \tag{2.13}$$

$$M(q^3)N(q)-q^2M(q)N(q^3)=\frac{f_1f_{18}^2}{f_4f_9f_{12}}, \tag{2.14}$$

$$M(q^5)N(q)-q^4M(q)N(q^5)=\frac{f_1f_6^2f_{30}}{f_2f_3f_{12}f_{20}}, \tag{2.15}$$

*and*

$$M(q)M(q^5)-q^6N(q)N(q^5)=\frac{f_5f_6f_{30}^2}{f_4f_{10}f_{15}f_{60}}. \tag{2.16}$$

A proof of identities (2.11)–(2.16) can be found in [15].

## 3. Main Results

In the present section, we deduce the relationship between the continued fraction $U(q)$ and $U(q^n)$ for $n=2,3,5,7,9,11,$ and $13$.

**Theorem 3.1.** *Let* $u:=U(q)$ *and* $v:=U(q^2)$. *Then*

$$u^2-v+2uv-u^2v+v^2=0.$$

***Proof.*** By (1.4), we have

$$U(q)=\frac{qN(q)}{M(q)}. \tag{3.1}$$

Dividing (2.11) by (2.12) and using (2.10), we get

$$\frac{M(q^2)N(q)+qM(q)N(q^2)}{M(q^2)N(q)-qM(q)N(q^2)}=\frac{\varphi(-q^3)}{\varphi(-q)}.$$

Squaring both sides of this identity, using (3.1) on its left-hand side and (2.2) on its right-hand side, and then dividing the equality obtained as a result by $4u$, we arrive at the required relation.

**Theorem 3.2** [11]. *Let* $u = U(q)$ *and* $v = U(q^3)$. *Then*

$$u^3 - v^3 + v^2 - v + 3uv - 3u^2v^2 + u^3v^2 - u^3v = 0.$$

***Proof.*** Dividing (2.14) by (2.13) and using (2.10), we find

$$\frac{M(q^3)N(q) - q^2M(q)N(q^3)}{M(q^3)N(q) + q^2M(q)N(q^3)} = \frac{\phi(-q^2)\phi(-q^3)\phi(-q^{18})}{\phi(-q^6)\phi(-q^9)\phi(-q^6)}.$$

Squaring both sides of this identity and using relation (3.1) on its left-hand side and the identity

$$\phi(q)\phi(-q) = \phi^2(-q^2)$$

on its right-hand side, we obtain

$$\left(\frac{U(q) - U(q^3)}{U(q) + U(q^3)}\right)^2 = \frac{\phi(q)}{\phi(q^3)}\frac{\phi(-q)}{\phi(-q^3)}\frac{\phi(-q^3)}{\phi(-q^9)}\frac{\phi(q^9)}{\phi(q^3)}.$$

Squaring again both sides of this identity, applying (2.1) and (2.2) on its right-hand side, and then factorizing by using Maple, we conclude that

$$8(-v^3 + v^2 - v + 3uv - 3u^2v^2 + u^3v^2 - u^3v + u^3)$$

$$\times (u^5v - u^4v^3 + u^4v^2 - 3u^4v + 5u^3v^3 - u^3v^2 - u^3v + u^3 - u^2v^5 + u^4v^2$$

$$+ u^2v^3 - 5u^2v^2 + 3v^4u - uv^3 + uv^2 - v^4) = 0.$$

Thus, it follows from the definition of $u$ and $v$ that

$$u = U(q) = 1 - q + q^5 - q^6 + q^7 + \dots \tag{3.2}$$

and

$$v = U(q^3) = 1 - q^3 + q^{15} - q^{18} + q^{21} + \dots.$$

By using (3.2) and the last relation in the same factors as above, we see that the first factor becomes

$$q^3(3 - 6q + 3q^2 - 3q^3 + 6q^4 + 3q^5 - 9q^6 + \dots)$$

and the second factor becomes

$$-q^3(10 - 22q + 15q^2 - 22q^3 + 33q^4 - 9q^5 + \dots).$$

Thus, the second factor does not vanish. Hence, by the identity Theorem, we must have

$$v^3 - v^2 + v - 3uv + 3u^2v^2 - u^3v^2 + u^3v - u^3 = 0.$$

Theorem 3.2 is proved.

**Theorem 3.3** [11]. *Let $u := U(q)$ and $v := U(q^5)$. Then*

$$uv^6 - u^6v^5 + 5u^5v^5 - 5u^4v^5 - 5uv^5 + 10v^4u^3 + 5uv^4$$

$$- 10u^4v^3 - 10u^2v^3 + 5u^5v^2 + 10u^3v^2 - 5u^5v - 5u^2v + 5uv - v + u^5 = 0.$$

***Proof.*** Dividing (2.15) by (2.16) and using (2.10), we find

$$\frac{M(q^5)N(q) - q^4M(q)N(q^5)}{M(q)M(q^5) - q^6N(q)N(q^5)} = \frac{q\psi(-q)}{\psi(-q^3)}\frac{\psi(-q^{15})}{\psi(-q^5)}.$$

Taking power 4 of both sides of this identity, employing relation (2.9) on its right-hand side, and then factorizing by using Maple, we conclude that

$$(u^2 + u^2v^2 + 1 - 4uv + v^2)(uv^6 - u^6v^5 + 5u^5v^5 - 5u^4v^5 - 5uv^5 + 10v^4u^3$$

$$+ 5uv^4 - 10u^4v^3 - 10u^2v^3 + 5u^5v^2 + 10u^3v^2 - 5u^5v - 5u^2v + 5uv - v + u^5) = 0.$$

Changing $q$ into $q^5$ in (3.2), we obtain

$$v = U(q^5) = 1 - q^5 + q^{25} - q^{30} + q^{35} + \ldots.$$

By using (3.2) and the last relation in the same factors as above, we see that the first factor becomes

$$q^2(2 - 4q^4 + 2q^5 - 4q^6 + 4q^8 - 2q^9 + 3q^{10} + \ldots)$$

and the second factor becomes

$$-q^7(40 - 40q + 20q^2 + 10q^3 - 55q^4 + 110q^5 - 180q^6 + \ldots).$$

Thus, the first factor does not vanish. Hence, by the identity theorem, we must have

$$uv^6 - u^6v^5 + 5u^5v^5 - 5u^4v^5 - 5uv^5 + 10v^4u^3 + 5uv^4 - 10u^4v^3$$

$$- 10u^2v^3 + 5u^5v^2 + 10u^3v^2 - 5u^5v - 5u^2v + 5uv - v + u^5 = 0.$$

Theorem 3.3 is proved.

**Theorem 3.4.** *Let* $u = U(q)$ *and* $v = U(q^7)$. *Then*

$$7u^3v^2 - 35u^3v^4 - 7u^3v - 7u^3v^3 + 35u^4v^3 - u^7$$
$$+ 7u^5v - 7u^6v + 7u^7v - 28u^5v^2 + 28u^6v^2 - 14v^2u^7 + 7v^3u^5 - 7v^3u^6$$
$$+ 7v^3u^7 - 35v^4u^5 + 7v^5u - 7v^5u^2 + 7v^5u^3 + 35v^5u^4 - 7v^5u^5 + 28v^5u^6$$
$$- 7v^5u^7 - 14v^6u + 28v^6u^2 - 28v^6u^3 + 7v^6u^5 - 28v^6u^6 + 7v^6u^7 + 7v^7u$$
$$- 7v^7u^2 + 7v^7u^3 - 7v^7u^5 + 14v^7u^6 - 7v^7u^7 + v^7u^8 - v^8u + v - 7uv$$
$$+ 7uv^2 + 14u^2v - 28u^2v^2 - 7uv^3 + 28u^2v^3 = 0.$$

***Proof.*** If

$$P = \frac{\phi(q)}{\phi(q^3)}$$

and

$$Q = \frac{\phi(q^7)}{\phi(q^{21})},$$

then

$$\left(\frac{Q}{P}\right)^4 - \left(\frac{P}{Q}\right)^4 + 14\left[\left(\frac{Q}{P}\right)^2 - \left(\frac{P}{Q}\right)^2\right]$$
$$= (PQ)^3 + \frac{27}{(PQ)^3} + 7\left(PQ + \frac{3}{PQ}\right)\left[1 - \left(\frac{P}{Q}\right)^2 - \left(\frac{Q}{P}\right)^2\right]. \tag{3.3}$$

We have deduced identity (3.3) on the lines remarked by N. D. Baruah [1]. Thus, by using (2.1) in (3.3), we arrive at the required result.

**Theorem 3.5.** *Let* $u = U(q)$ *and* $v = U(q^9)$. *Then*

$$9u^4v + 126v^4u - 90uv^3 + 27v^8u^3 + v - 9vu + 45v^2u + 378u^7v^5$$
$$+ 135u^8v^4 + 54u^8v^2 - 9u^8v + 18u^7v + 558v^4u^6 - 558v^5u^3 + 561v^4u^3$$
$$- 10u^9v^2 - 333u^6v^3 + 4u^9v - 561u^6v^5 + 16u^9v^3 - 19u^9v^4 + 369v^5u^2$$
$$- 99u^5v^2 - 378v^4u^2 + 135v^7u^2 + 99uv^6 + 16u^9v^5 + 333u^3v^6$$

$$+ 243u^7v^3 - 243u^2v^6 - 126u^8v^5 - 27u^6v - 54v^7u + 9v^8u - 135u^7v^2$$
$$+ 10v^3 - 198v^6u^4 + 369u^4v^5 - 16v^6 + 19v^5 - 16v^4 - u^9 - 4v^8$$
$$+ 10v^7 - 243v^4u^4 - 153v^2u^2 + 99v^7u^4 + 297v^3u^2 + 9u^8v^8 + 243u^5v^5$$
$$- 18v^8u^2 - 297v^6u^7 + 30u^6v^8 + 153v^7u^7 - 30vu^3 - 165v^7u^3$$
$$+ 198u^5v^3 - 99u^8v^3 - 423v^3u^3 - 9u^4v^8 - 369u^7v^4 + 165u^6v^2$$
$$+ 423u^6v^6 - 153u^5v^6 - 171u^6v^7 - 369u^5v^4 - 9u^5v^8 + 54u^5v^7$$
$$+ 27vu^2 + 90u^8v^6 + v^9 + 9u^5v - 10u^9v^6 - 45u^8v^7 + 171v^2u^3$$
$$+ 153v^3u^4 + 4u^9v^7 - 4v^2 - 27u^7v^8 - 54u^4v^2 - u^9v^8 - 135v^5u = 0.$$

***Proof.*** Let $w = U(q^3)$. Then, by Theorem 3.2, we get

$$u^3 - w^3 + w^2 - w + 3uw - 3u^2w^2 + u^3w^2 - u^3w = 0. \tag{3.4}$$

Changing $q$ into $q^3$ in relation (3.4), we obtain

$$w^3 - v^3 + v^2 - v + 3wv - 3w^2v^2 + w^3v^2 - w^3v = 0. \tag{3.5}$$

Further, eliminating $w$ between (3.4) and (3.5), we deduce the required result.

**Theorem 3.6.** *Let* $u = U(q)$ *and* $v = U(q^{11})$. *Then*

$$-363v^7u^{10} - 528v^{10}u^8 + 11v^{10}u^{11} - 1089v^9u^8 + 1463v^4u^8 - 1012v^7u^8$$
$$+ 11v^{11}u^2 + vu - 968v^4u^4 + 44v^3u - 374v^3u^2 + 77v^4u$$
$$- 528v^4u^2 - 11vu^2 - 363v^5u^2 + 55v^7u^{11} + v^{11}u^{11}$$
$$+ 759v^7u^9 + 55v^5u + 55v^11u^7 - 759v^3u^7 - 55v^7u$$
$$+ 363v^7u^2 - 77v^8u + 528v^8u^2 + 176v^{10}u^2$$
$$+ 44v^{11}u^9 + 363v^2u^5 - 44vu^9 - 11v^2u^{11} - 11v^{10}u$$
$$+ 528v^{10}u^4 + 528v^4u^{10} + 374v^9u^2 - 44v^9u$$

$$- 110v^2u^2 + 11v^2u + 77vu^8 + 374v^3u^{10} - 363v^2u^7$$

$$- 77vu^4 + 11vu^{10} + 1089v^4u^3 - 374v^9u^{10}$$

$$+ 44vu^3 - 759v^9u^5 + 55vu^5 + 759v^5u^3$$

$$+ 363v^5u^{10} + 1012v^5u^8 - 1012v^5u^4 - 55vu^7 + 1496v^5u^5$$

$$- 759v^5u^9 - 528v^2u^4 - 704v^5u^7 - 55v^5u^{11} - 968v^8u^8$$

$$+ 1463v^8u^4 + 1089v^3u^8 - 528v^8u^{10} - 1089v^3u^4$$

$$- 1089v^8u^3 - 1012v^8u^5 + 1496v^7u^7 + 77v^8u^{11}$$

$$+ 1089v^8u^9 - 363v^{10}u^5 + 1012v^8u^7 + u^{12} - 374v^2u^9$$

$$+ 1023v^3u^3 - 110v^{10}u^{10}1012v^7u^4 - 704v^7u^5 - 77v^{11}u^8$$

$$+ 1012v^4u^5 + 11vu^{11} - 44v^{11}u^3 - 1012v^4u^7 + 924v^6u^6$$

$$+ 1089v^9u^4 + 1023v^9u^9 - 759v^7u^3 + 759v^3u^5 + 44v^9u^{11}$$

$$+ 528v^2u^8 + 759v^9u^7 - 803v^3u^9 - 44v^3u^{11} + 363v^{10}u^7$$

$$+ 1089v^4u^9 + 11v^{11}u + 374v^2u^3 - 77v^4u^{11} + 77v^{11}u^4$$

$$+ 374v^{10}u^9 - 55v^{11}u^5 + v^{12} - 374v^{10}u^3 - 11v^{11}u^{10} - 803v^9u^3 + 176v^2u^{10} = 0.$$

***Proof.*** Let

$$P = \frac{\varphi(q)}{\varphi(q^3)}$$

and

$$Q = \frac{\varphi(q^{11})}{\varphi(q^{33})}.$$

Then it follows from [18] that

$$(PQ)^5+\frac{3^5}{(PQ)^5}-11\left[(PQ)^3+\frac{3^3}{(PQ)^3}\right]+308\left[PQ+\frac{3}{PQ}\right]$$
$$=\left(\frac{P}{Q}\right)^6+\left(\frac{Q}{P}\right)^6+22\left[\left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4\right]\left[3-\left(PQ+\frac{3}{PQ}\right)\right]$$
$$+11\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]\left[(PQ)^3+\frac{3^3}{(PQ)^3}-15\left(PQ+\frac{3}{PQ}\right)+45\right]+924.$$

By using (2.1) in the last relation, we arrive at the required result.

**Theorem 3.7.** *Let $u=U(q)$ and $v=U(q^{13})$. Then*

$$-8346u^8v^5+611u^{11}v^{12}+6318u^4v^8+8723u^8v^6-8346u^5v^8-78u^7v^5$$
$$+910u^{10}v^7+1508u^8v^2-780u^8v^7+8476u^8v^9-10322u^4v^9-5148u^8v^8$$
$$-481u^3v^{12}-7228u^8v^{10}+10894u^5v^9+611u^3v^2-208u^6v^{13}+910u^7v^{10}$$
$$-10322u^5v^{10}-143u^{12}v^{12}+6409u^3v^9-1911u^5v^{12}+1313u^{10}v^2$$
$$-832u^7v^{11}-5317u^3v^{10}-130u^7v^{12}-2184u^{11}v^{11}-182u^{13}v^9+2652u^3v^{11}$$
$$-5148u^6v^6-5317u^{10}v^3-91u^{13}v^4+1508u^6v^{12}+78uv^7+8723u^6v^8$$
$$+4706u^6v^3+1846u^5v^2+1508u^2v^8+v^{14}+9607u^{10}v^4-5317u^4v^{11}$$
$$-182uv^5+78u^6v-1170u^6v^2-2184u^3v^3-7228u^4v^6-6331u^{11}v^9$$
$$+195u^9v+10894u^9v^5+6409u^9v^3-7228u^{10}v^8+6318u^{10}v^6+78u^7v^{13}$$
$$+4706u^3v^6-780u^6v^7+13u^{13}v^{12}-130u^2v^7+52u^2v^{13}-1911u^9v^2$$
$$+6318u^8v^4+78uv^6+4407u^{11}v^{10}+1846u^2v^5-8346u^6v^9-1768u^7v^7$$
$$-8346u^9v^6-832u^7v^3-10322u^{10}v^5-7631u^4v^4-3926u^8v^3-780u^7v^8$$
$$+910u^7v^4-130u^7v^2+78u^7v+10257u^4v^5+u^{14}-832u^3v^7-78u^7v^9$$
$$-78u^9v^7+10257u^{10}v^9+8476u^9v^8-780u^7v^6-12909u^9v^9-7631u^{10}v^{10}$$

$$+ 846u^{12}v^{9} - u^{13}v^{13} + 156u^{13}v^{10} - 10322u^{9}v^{4} + 4407u^{4}v^{3} + 13uv^{2}$$
$$+ 10257u^{9}v^{10} + 6409u^{11}v^{5} - 6331u^{5}v^{3} - 1378u^{4}v^{2} + 13u^{2}v + 4407u^{10}v^{11}$$
$$- 3926u^{11}v^{6} - 3926u^{3}v^{8} - 143u^{2}v^{2} - 1911u^{12}v^{5} + 6409u^{5}v^{11} + 8476u^{6}v^{5}$$
$$+ 156uv^{4} - 65u^{3}v + 1508u^{12}v^{6} - 208uv^{8} + 13u^{12}v^{13} - 65u^{11}v^{13} + 156u^{10}v^{13}$$
$$+ 195u^{13}v^{5} - 1378u^{10}v^{12} + 1313u^{4}v^{12} - 91u^{4}v^{13} - 130u^{12}v^{7} - 3926u^{6}v^{11}$$
$$- 832u^{11}v^{7} - 65uv^{3} - 208u^{13}v^{6} - 1170u^{12}v^{8} + 9607u^{4}v^{10} - 13u^{3}v^{13}$$
$$+ 6318u^{6}v^{10} + 78u^{13}v^{8} + 611u^{12}v^{11} + 156u^{4}v + 611u^{2}v^{3} + 78u^{13}v^{7}$$
$$+ 4706u^{11}v^{8} - 1378u^{2}v^{4} + 10257u^{5}v^{4} - 208u^{8}v - 26u^{13}v - 7228u^{6}v^{4}$$
$$- 91u^{10}v + 52u^{12}v + 4407u^{3}v^{4} - 13u^{11}v + 8476u^{5}v^{6} - 481u^{11}v^{2} - 1378u^{12}v^{10}$$
$$- 1170u^{8}v^{12} + 78u^{8}v^{13} + 4706u^{8}v^{11} - 182u^{5}v - 13uv^{11} + 195uv^{9} - 91uv^{10}$$
$$- 65u^{13}v^{11} - uv - 39u^{2}v^{12} - 481u^{2}v^{11} - 26uv^{13} - 1170u^{2}v^{6} + 52uv^{12}$$
$$- 1911u^{2}v^{9} + 1846u^{9}v^{12} + 910u^{4}v^{7} - 6331u^{9}v^{11} - 13u^{13}v^{3} + 2652u^{11}v^{3}$$
$$- 481u^{12}v^{3} - 182u^{9}v^{13} + 52u^{13}v^{2} + 195u^{5}v^{13} + 1313u^{2}v^{10} - 39u^{12}v^{2}$$
$$- 6331u^{3}v^{5} - 5317u^{11}v^{4} + 1313u^{12}v^{4} - 12909u^{5}v^{5} - 78u^{5}v^{7} = 0.$$

***Proof.*** Let

$$P = \frac{\varphi(q)}{\varphi(q^3)} \quad \text{and} \quad Q = \frac{\varphi(q^{13})}{\varphi(q^{39})}.$$

Then it follows from [18] that

$$\left(\frac{Q}{P}\right)^7 + \left(\frac{P}{Q}\right)^7 + 13\left[\left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6\right] - 26\left[\left(\frac{P}{Q}\right)^5 + \left(\frac{Q}{P}\right)^5\right]$$
$$- 13\left[3(PQ)^2 + \frac{27}{(PQ)^2} + 10\right]\left[\left(\frac{Q}{P}\right)^4 + \left(\frac{P}{Q}\right)^4\right] + 13\left[10(PQ)^2 + \frac{90}{(PQ)^2} + 68\right]$$

$$\times\left[\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3\right]+13\left[10(PQ)^2+\frac{90}{(PQ)^2}+68\right]\left[\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3\right]$$

$$+13\left[(PQ)^4+\frac{81}{(PQ)^4}-20\left((PQ)^2+\frac{9}{(PQ)^2}\right)-115\right]\left[\left(\frac{Q}{P}\right)^2+\left(\frac{P}{Q}\right)^2\right]$$

$$-13\left[(PQ)^4+\frac{81}{(PQ)^4}-10\left((PQ)^2+\frac{9}{(PQ)^2}\right)-131\right]\left[\left(\frac{Q}{P}\right)+\left(\frac{P}{Q}\right)\right]$$

$$=(PQ)^6+\frac{729}{(PQ)^6}-26\left[(PQ)^4+\frac{81}{(PQ)^4}\right]+169\left[(PQ)^2+\frac{9}{(PQ)^2}\right]+832.$$

Finally, by using (2.1) in the last identity, we arrive at the required result.


## Acknowledgements

The authors are thankful to the DST, New Delhi, for awarding the research project [No. SR/S4/MS:517/08] under which the present work has been done. Further, the authors thank the referee for the comments aimed at improving the manuscript.


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