ON SPECTRA OF VARIANTS OF THE CORONA OF TWO GRAPHS AND SOME NEW EQUIENERGETIC GRAPHS

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Abstract

Let $G$ and $H$ be two graphs. The join $G \vee H$ is the graph obtained by joining every vertex of $G$ with every vertex of $H$. The corona $G \circ H$ is the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and joining the $i$-th vertex of $G$ to every vertex in the $i$-th copy of $H$. The neighborhood corona $G \star H$ is the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and joining the neighbors of the $i$-th vertex of $G$ to every vertex in the $i$-th copy of $H$. The edge corona $G \diamond H$ is the graph obtained by taking one copy of $G$ and $|E(G)|$ copies of $H$ and joining each terminal vertex of the $i$-th edge of $G$ to every vertex in the $i$-th copy of $H$. Let $G_1$, $G_2$, $G_3$ and $G_4$ be regular graphs with disjoint vertex sets. In this paper we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \star G_3)$, $(G_1 \vee G_2) \cup (G_2 \star G_3) \cup (G_1 \star G_4)$, $(G_1 \vee G_2) \cup (G_1 \star G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3)$, $(G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, $(G_1 \circ G_2)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3)$ and $(G_1 \circ G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$.

As an application, we show that there exist some new pairs of equienergetic graphs on $n$ vertices for all $n \geq 11$.

Keywords: spectrum, corona, neighbourhood corona, edge corona, energy of a graph, equienergetic graphs.

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1. Introduction

Throughout this paper we consider only undirected simple graphs (i.e., graphs with no loops and multiple edges). Let $G$ be a graph on $n$ vertices. The eigenvalues of the adjacency matrix of $G$, denoted by $\lambda_i(G)$, $i = 1, 2, \ldots, n$, are
the eigenvalues of the graph $G$ and $\sigma(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G))$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ is the adjacency spectrum of $G$ \[8\]. The energy $E(G)$ is the sum of all absolute values of eigenvalues of $G$. The concept of energy of a graph was introduced by Gutman \[12\] with an application to chemistry (Huckel molecular orbital approximation for the total $\pi$-electron energy \[14\]). The energy and various forms of energy of a graph $G$ has been extensively studied by many mathematicians and some of their works can be found in \[1, 2, 3, 5, 13, 15, 19, 21, 28, 27\] and references therein. Two graphs $G_1$ and $G_2$ of the same order are said to be equienergetic if $E(G_1) = E(G_2)$. The energy and various forms of energy of a graph $G$ has been extensively studied by many mathematicians and some of their work can be found in \[14\]). The energy and various forms of energy of a graph $G$ has been extensively studied by many mathematicians and some of their works can be found in \[4, 7, 10, 24\]. The neighborhood corona and edge corona was introduced in \[17\], respectively. Complete description of the spectrum of neighborhood corona and edge corona of two graphs are given in \[17, 23\] and references therein.

Later Liu et al. \[22\] and Ramane, Walikar \[26\] have independently proved that there exists a pair of equienergetic graphs on $n$ vertices for all $n \geq 9$. Studies on equienergetic graphs can be found in \[6, 11, 18, 22, 25, 26, 29\] and references therein.

The corona of two graphs was first introduced by Frucht and Harary in \[10\]. Barik et al. \[4\] provided a complete description of the spectrum of corona $G_1 \circ G_2$ using the spectrum of $G_1$ and $G_2$. More about the spectrum of corona can be found in \[4, 7, 10, 24\]. The neighborhood corona and edge corona was introduced in \[17\] and in \[16\], respectively. Complete description of the spectrum of neighborhood corona and edge corona of two graphs are given in \[17, 23\] and \[16\], respectively.

Motivated by the above works, in this paper we compute the spectrum of $(G_1 \vee G_2) \cup (G_1 \ast G_3)$, $(G_1 \vee G_2) \cup (G_2 \ast G_3) \cup (G_1 \ast G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \ast G_3)$, $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ and $(G_1 \vee G_2) \cup (G_2 \ast G_3) \cup (G_1 \circ G_4)$, when $G_1$, $G_2$, $G_3$ and $G_4$ are regular graphs. Here the graphs $G_1$, $G_2$, $G_3$ and $G_4$ have disjoint vertex sets. As an application of our results we construct some new pairs of equienergetic graphs on $n$ vertices for all $n \geq 11$. Our method of construction and proofs are entirely different from the methods given in \[18, 22, 26\].

2. Preliminaries

In this section, we give some definitions and lemmas which are useful to prove our main results.

**Definition** \[10\]. Let $G_1$ and $G_2$ be two graphs on $n$ and $m$ vertices, respectively. The corona $G_1 \circ G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one
copy of $G_1$ and $n$ copies of $G_2$, and then joining the $i$-th vertex of $G_1$ to every vertex in the $i$-th copy of $G_2$.

**Definition** [16]. Let $G_1$ and $G_2$ be two graphs on $n_1$ and $n_2$ vertices, $m_1$ and $m_2$ edges, respectively. The edge corona $G_1 \circ G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ and $m_1$ copies of $G_2$, and then joining two end vertices of the $i$-th edge of $G_1$ to every vertex in the $i$-th copy of $G_2$.

**Definition** [17]. Let $G_1$ and $G_2$ be two graphs on $n$ and $m$ vertices, respectively. The neighborhood corona $G_1 \star G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ and $n$ copies of $G_2$, and then joining each neighbor of $i$-th vertex of $G_1$ to every vertex in the $i$-th copy of $G_2$.

**Definition** [8]. Let $A = (a_{ij})$ be an $n \times m$ matrix, $B = (b_{ij})$ be a $p \times q$ matrix. Then the Kronecker product $A \otimes B$ of $A$ and $B$ is the $np \times mq$ matrix obtained by replacing each entry $a_{ij}$ of $A$ by $a_{ij}B$.

**Lemma 1** [8]. If $M$, $N$, $P$, $Q$ are matrices with $M$ being a non-singular matrix, then

$$\det(MN - PQ) = |M||Q - PM^{-1}N|.$$  

**Lemma 2** [26]. Let $N_1$ and $N_2$ be two graphs as shown in Figure 1. Then the line graph $L(N_1)$ of $N_1$ and the line graph $L(N_2)$ of $N_2$ are non cospectral and equienergetic.

**Figure 1**

**Lemma 3** [8]. The following cubic regular graphs with ten vertices are equienergetic.
3. Spectra of \((G_1 \lor G_2) \cup (G_1 \ast G_3)\) and \((G_1 \lor G_2) \cup (G_2 \ast G_3) \cup (G_1 \ast G_4)\)

In this section, we compute the spectrum of \((G_1 \lor G_2) \cup (G_1 \ast G_3)\) and \((G_1 \lor G_2) \cup (G_2 \ast G_3) \cup (G_1 \ast G_4)\), where \(G_1, G_2, G_3\) and \(G_4\) are regular graphs on \(n, m, l\) and \(p\) vertices, respectively.

**Theorem 4.** Let \(G_i\) be \(r_i\)-regular graphs \((i = 1, 2, 3)\). Suppose \(\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)\), \(\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)\) and \(\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)\) are the adjacency spectrum of \(G_1, G_2\) and \(G_3\), respectively. Then the adjacency spectrum of \(G = (G_1 \lor G_2) \cup (G_1 \ast G_3)\) is

\[
\sigma(G) = \begin{pmatrix}
\gamma_i & \mu_j & 1 + \frac{\sqrt{4l\lambda_k^2 + (\lambda_k - r_3)^2}}{2} & x_t \\
n & 1 & 1 & 1
\end{pmatrix},
\]

where \(i = 2\) to \(l\), \(j = 2\) to \(m\), \(k = 2\) to \(n\), \(t = 1, 2, 3\). Also, the entries in the first row are the eigenvalues with multiplicity written below, and \(x_t\)'s are the roots of the polynomial \((x - r_2) ((x - r_1)(x - r_3) - lr_1^2) - nm(x - r_3)\).

**Proof.** With suitable labelling of the vertices of \(G\), the adjacency matrix \(A(G)\) can be formulated as follows:

\[
A(G) = \begin{pmatrix}
I_n \otimes A(G_3) & 0 & A(G_1) \otimes e \\
0 & A(G_2) & J \\
A(G_1) \otimes e^T & J^T & A(G_1)
\end{pmatrix},
\]

where \(e^T = (1, 1, \ldots, 1)\), \(I_n\) is the identity matrix of order \(n\) and \(J\) is the \(m \times n\) matrix with all its entries are 1.
Since $A(G_3)$ is a real symmetric matrix, $A(G_3)$ is orthogonally diagonalizable. Let $A(G_3) = PDP^T$, where $PP^T = I_l$ and $D = diag(\gamma_1, \ldots, \gamma_l)$. Then

$$A(G) = \begin{pmatrix} I_n \otimes PDP^T & 0 & A(G_1) \otimes e^T \\ 0 & A(G_2) & J \\ A(G_1) \otimes e^T & J^T & A(G_1) \end{pmatrix}$$

$$= \begin{pmatrix} I_n \otimes P & 0 & 0 \\ 0 & I_n \otimes D & 0 & A(G_1) \otimes P^T e^T \\ 0 & A(G_1) \otimes e^T P & J^T & A(G_1) \end{pmatrix} = \begin{pmatrix} I_n \otimes P^T & 0 \\ 0 & I_n \otimes D & 0 & A(G_1) \otimes \sqrt{\lambda_1} e_1^T \\ 0 & A(G_1) \otimes \sqrt{\lambda_1} e_1^T & J^T & A(G_1) \end{pmatrix}$$

Let $B = \begin{pmatrix} I_n \otimes D & 0 & A(G_1) \otimes \sqrt{\lambda_1} e_1^T \\ 0 & A(G_2) & J \\ A(G_1) \otimes \sqrt{\lambda_1} e_1^T & J^T & A(G_1) \end{pmatrix}$.

Then by the above equation we have

$$(2) \quad |xI - A(G)| = |xI - B|.$$
Again as \( A(G_1) \) and \( A(G_2) \) are orthogonally diagonalizable, one can easily see that the \( M_i \) is the same as

\[
M'_i = \begin{pmatrix}
(x - r_3)I_n & 0 & -\sqrt{m} \text{diag}(\lambda_1, \ldots, \lambda_n) \\
0 & \text{diag}(x - \mu_1, \ldots, x - \mu_m) & -\sqrt{mn}J' \\
-\sqrt{m} \text{diag}(\lambda_1, \ldots, \lambda_n) & -\sqrt{mn}J'^T & \text{diag}(x - \lambda_1, \ldots, x - \lambda_n)
\end{pmatrix},
\]

where \( J' \) is the matrix obtained by replacing every entries of \( J \) except the first diagonal entry by 0. Now by (1), we have

\[
M'_i = (x - r_3)^n
\]

(5) \[
\times \begin{pmatrix}
\text{diag}(x - r_2, x - \mu_2, \ldots, x - \mu_m) & -\sqrt{mn}J' \\
-\sqrt{mn}J'^T & \text{diag}(x - \lambda_1 - l\lambda_2/(x - r_3), \ldots, x - \lambda_n - l\lambda_n^2/(x - r_3))
\end{pmatrix}
\]

Applying Laplace method along \( 2, \ldots, m, m + 2, \ldots, m + n \) columns in the above determinant we see that the only non zero \( m + n - 2 \times m + n - 2 \) minor is \( \text{diag}(x - \mu_2, \ldots, x - \mu_m, x - \lambda_2 - l\lambda_2/(x - r_3), \ldots, x - \lambda_n - l\lambda_n^2/(x - r_3)) \) and the complementary minor is

\[
M_1 = \begin{vmatrix}
x - \mu_2 & -\sqrt{mn} \\
-\sqrt{mn} & x - \lambda_2 - l\lambda_2^2/(x - r_3)
\end{vmatrix}.
\]

And so by (2), (3), (4), (5) and from the above equation the result follows. \( \blacksquare \)

**Corollary 5.** Let \( G_i \) be \( r_i \)-regular graphs \((i = 1, 2)\). Then

\[
E(G_1 \lor G_2 \cup G_1 \ast lK_1) = \sqrt{4l + 1}E(G_1) + E(G_2) - r_1(\sqrt{4l + 1} - 1) - 2x_0,
\]

where \( x_0 \) is the negative root of the polynomial \((x - r_2)\left((x - r_1) x - h_r^2\right) - nmx \).

**Remark 6.** Corollary 5 is a generalization of Theorem 1 in [18]. In fact putting \( r_1 = r, n = p, r_2 = 0, m = k, r_3 = 0, l = 1 \) in Corollary 5, we obtain Theorem 1 due to Indulal and Vijayakumar [18].

**Corollary 7.** Let \( G_i \) \((i = 1, 2)\) be equienergetic regular graphs of the same degree and \( H_i \) \((i = 1, 2)\) be equienergetic regular graphs of the same degree. Then

\[
E(G_1 \lor H_1 \cup G_1 \ast lK_1) = E(G_2 \lor H_2 \cup G_2 \ast lK_1).
\]

Now we construct some new pairs of equienergetic graphs using Corollary 7.

**Theorem 8.** There exists a pair of equienergetic graphs on \( n \) vertices for all \( n \geq 18 \).
Proof. From Lemma 2 we have the line graphs $L(N_1)$ and $L(N_2)$ are equienergetic. Now by Corollary 7 it is clear that the graphs $(L(N_1) \vee K_m) \cup (L(N_1) \ast K_1)$ and $(L(N_2) \vee K_m) \cup (L(N_2) \ast K_1)$, both of order $18 + m$ ($m = 0, 1, \ldots$), are equienergetic. This completes the proof of the theorem.

Theorem 9. There exists a pair of equienergetic graphs on $n$ vertices for all $n \geq 20$.

Proof. From Lemma 3 and Corollary 7 it is clear that the graphs $(T_1 \vee K_m) \cup (T_1 \ast K_1)$ and $(T_2 \vee K_m) \cup (T_2 \ast K_1)$, both of order $20 + m$ ($m = 0, 1, \ldots$), are equienergetic.

Theorem 10. There exists a pair of equienergetic graphs on $n$ vertices for all $n \geq 13$.

Proof. Case 1. $n = 9 + 2m$ ($m = 2, 3, \ldots$). For $n = 9 + 2m$ ($m = 2, 3, \ldots$), the graphs $(K_m \vee L(N_1)) \cup (K_m \ast K_1)$ and $(K_m \vee L(N_2)) \cup (K_m \ast K_1)$ both are of order $9 + 2m$ ($m = 2, 3, \ldots$). Now, Corollary 7 implies that these two graphs are equienergetic.

Case 2. $n = 10 + 2m$ ($m = 2, 3, \ldots$). For $n = 10 + 2m$ ($m = 2, 3, \ldots$), the graphs $(K_m \vee T_1) \cup (K_m \ast K_1)$ and $(K_m \vee T_2) \cup (K_m \ast K_1)$ both are of order $10 + 2m$ ($m = 2, 3, \ldots$). Now, Corollary 7 implies that these two graphs are equienergetic.

As the proof of the following theorem is similar to that of Theorem 4, we omit the details.

Theorem 11. Let $G_i$ be $r_i$-regular graphs ($i = 1, 2, 3, 4$). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of $G_1$, $G_2$, $G_3$ and $G_4$, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \ast G_3) \cup (G_1 \ast G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j \\ m & n \end{pmatrix} \begin{pmatrix} \lambda_k + r_4 \pm \sqrt{4p\lambda_k^2 + (\lambda_k - r_4)^2} \\ 1 \end{pmatrix} / 2$$

$$\begin{pmatrix} \mu_s + r_3 \pm \sqrt{4l\mu_s^2 + (\mu_s - r_3)^2} \\ 1 \end{pmatrix} / 2$$

$$x_t,$$

where $i = 2$ to $l$, $j = 2$ to $p$, $k = 2$ to $n$, $s = 2$ to $m$, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_t$’s are the roots of the polynomial

$$(x - r_1)(x - r_4) - pr_1^2 \left( (x - r_2)(x - r_3) - lr_2^2 \right) - nm(x - r_3)(x - r_4).$$
Corollary 12. Let $G_i$ be $r_i$-regular graphs $(i = 1, 2)$. Then
\[
E(G_1 \vee G_2 \cup G_2 \star lK_1 \cup G_1 \star pK_1) = \sqrt{4p+1}E(G_1) + \sqrt{4l+1}E(G_2) - r_2(\sqrt{4l+1} - 1) \\
- r_1(\sqrt{4p+1} - 1) - 2x_0 - 2x_1,
\]
where $x_0$ and $x_1$ are the negative roots of the polynomial
\[
x^4 - (r_1 + r_2)x^3 + (-r_1^2p - lr_2^2 + r_1r_2 - mn)x^2 + (r_1^2r_2p + r_1r_2^2l)x + r_1^2r_2^2lp.
\]

Corollary 13. Let $G_1$, $G_2$ be equienergetic regular graphs of the same degree and $H_1$, $H_2$ be equienergetic regular graphs of the same degree. Then
\[
E(G_1 \vee H_1 \cup H_1 \star lK_1 \cup G_1 \star pK_1) = E(G_2 \vee H_2 \cup H_2 \star lK_1 \cup G_2 \star pK_1).
\]

4. Spectra of $(G_1 \vee G_2) \cup (G_1 \circ G_3)$ and $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$

In this section, we simply state some theorems (as the proofs are quite analogous to the proof of Theorem 4) which gives the spectrum of $(G_1 \vee G_2) \cup (G_1 \circ G_3)$ and $(G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$, where $G_1$, $G_2$, $G_3$, and $G_4$ are regular graphs on $n$, $m$, $l$, and $p$ vertices, respectively.

Theorem 14. Let $G_i$ be $r_i$-regular graphs $(i = 1, 2, 3)$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$ and $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_t)$ are the adjacency spectrum of $G_1$, $G_2$, and $G_3$, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_1 \circ G_3)$ is
\[
\sigma(G) = \left(\begin{array}{cccc} 
\gamma_i & \mu_j & x_t & 0 \\
-1 & -1 & 1 & 1 \\
\lambda_k + r_3 \pm \sqrt{4l + (\lambda_k - r_3)^2} / 2 & x_t & 1 & 1 \\
\end{array}\right),
\]
where $i = 2$ to $l$, $j = 2$ to $m$, $k = 2$ to $n$, $t = 1, 2, 3$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_t$'s are the roots of the polynomial $(x - r_2)((x - r_1)(x - r_3) - l) - nm(x - r_3)$.

Theorem 15. Let $G$ be an $r$-regular graph of order $m$. Then
\[
E(K_n \vee G \cup K_n \circ lK_1) = E(G) + (n - 1)\sqrt{4l+1} - 2x_0 + n - 1,
\]
where $x_0$ is the negative root of the polynomial $(x - r)(x(x - (n - 1)) - l) - nmx$.

Corollary 16. Let $G$ and $H$ be equienergetic regular graphs of the same degree. Then
\[
E(K_n \vee G \cup K_n \circ lK_1) = E(K_n \vee H \cup K_n \circ lK_1).
\]
Theorem 17. Let $G$ be an $r$-regular graph of order $m$. Then

$$E(nK_1 \vee G \cup nK_1 \circ lK_1) = E(G) + (n - 1)\sqrt{4l} - 2x_0,$$

where $x_0$ is the negative root of the polynomial $(x - r) (x^2 - l) - nm x$.

Corollary 18. Let $G$ and $H$ be equienergetic regular graphs of the same degree. Then

$$E(nK_1 \vee G \cup nK_1 \circ lK_1) = E(nK_1 \vee H \cup nK_1 \circ lK_1).$$

Now we construct some new pairs of equienergetic graphs using Corollary 16.

Theorem 19. There exists a pair of equienergetic graphs on $n$ vertices for all $n \geq 11$.

Proof. Case 1. $n = 9 + 2m$ ($m = 1, 2, \ldots$). For $n = 9 + 2m$ ($m = 1, 2, \ldots$), the graphs $(K_m \vee L(N_1)) \cup (K_m \circ K_1)$ and $(K_m \vee L(N_2)) \cup (K_m \circ K_1)$ both are of order $9 + 2m$ ($m = 1, 2, \ldots$). Now, Corollary 16 implies that these two graphs are equienergetic.

Case 2. $n = 10 + 2m$ ($m = 1, 2, \ldots$). For $n = 10 + 2m$ ($m = 1, 2, \ldots$), the graphs $(K_m \vee T_1) \cup (K_m \circ K_1)$ and $(K_m \vee T_2) \cup (K_m \circ K_1)$ both are of order $10 + 2m$ ($m = 1, 2, \ldots$). Now, Corollary 16 implies that these two graphs are equienergetic. □

Theorem 20. Let $G_i$ be $r_i$-regular graphs ($i = 1, 2, 3, 4$). Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_s)$ are the adjacency spectrum of $G_1, G_2, G_3$ and $G_4$, respectively. Then the adjacency spectrum of $G = (G_1 \vee G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)$ is

$$\sigma(G) = \begin{pmatrix} \gamma_i & \eta_j & \lambda_k + r_4 \pm \sqrt{4p + (\lambda_k - r_4)^2} / 2 \\ m & n & 1 \\ \mu_s + r_3 \pm \sqrt{4l + (\mu_s - r_3)^2} / 2 & x_t & 1 \end{pmatrix},$$

where $i = 2$ to $l$, $j = 2$ to $p$, $k = 2$ to $n$, $s = 2$ to $m$, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_t$’s are the roots of the polynomial

$$(x - r_1) (x - r_4 - p) ((x - r_2) (x - r_3) - l) - nm (x - r_3) (x - r_4).$$
In Theorems 21 and 25 of this section, we compute the spectrum of \((G_1 \lor G_2) \cup (G_1 \circ G_3)\) and \((G_1 \lor G_2) \cup (G_2 \circ G_3) \cup (G_1 \circ G_4)\), where \(G_1, G_2, G_3\) and \(G_4\) are regular graphs on \(n, m, l\) and \(p\) vertices, respectively. Proofs of these theorems are not given as they are similar to the proof of Theorem 4.

**Theorem 21.** Let \(G_i\) be \(r_i\)-regular graphs \((i = 1, 2, 3)\) and \(r_1 \geq 2\). Suppose \(\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)\), \(\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m)\) and \(\sigma(G_3) = (\gamma_1 = r_2, \gamma_2, \ldots, \gamma_l)\) are the adjacency spectrum of \(G_1\), \(G_2\) and \(G_3\), respectively. Then the adjacency spectrum of \(G = (G_1 \lor G_2) \cup (G_1 \circ G_3)\) is

\[
\sigma(G) = \left[ \begin{array}{c c c}
\gamma_1 & r_3 & \mu_j \\
(\gamma_1 n/2) & (r_3 n/2) & 1 \\
1 & 1 & 1
\end{array} \right],
\]

where \(i = 2\) to \(l\), \(j = 2\) to \(m\), \(k = 2\) to \(n\), \(t = 1, 2, 3\). Also, the entries in the first row are the eigenvalues with multiplicity written below, and \(x_t\)'s are the roots of the polynomial \((x - r_2) (x - r_1) (x - r_3) - 2lr_1 - nm (x - r_3)\).

**Theorem 22.** Let \(G\) be an \(r\)-regular graph of order \(m\). Then

\[
E(K_n \lor G \cup K_n \circ lK_1) = E(G) + (n - 1)((\sqrt{4l + n - 2} + 1) - 2x_0),
\]

where \(x_0\) is the negative root of the polynomial

\[
x^3 - (n - 1 + r)x^2 + ((n - 1)r - 2n - 1)l - mn)x + 2(n - 1)rl.
\]

**Corollary 23.** Let \(G\) and \(H\) be equienergetic regular graphs of the same degree. Then

\[
E(K_n \lor G \cup K_n \circ lK_1) = E(K_n \lor H \cup K_n \circ lK_1).
\]

Now we construct some new pairs of equienergetic graphs using Corollary 23.

**Theorem 24.** There exists a pair of equienergetic graphs on \(n\) vertices for all \(n \geq 15\).

**Proof.** Case 1. Let \(n = 9 + 2m\) \((m = 3, 4, \ldots)\) and \(C_m\) be the cycle of length \(m\). Then, by Corollary 23 and Lemma 2 the graphs \((C_m \lor L(N_1)) \cup (C_m \circ K_1)\) and \((C_m \lor L(N_2)) \cup (C_m \circ K_1)\), both of order \(9 + 2m\) \((m = 3, 4, \ldots)\), are equienergetic.

Case 2. \(n = 10 + 2m\) \((m = 3, 4, \ldots)\). For \(n = 10 + 2m\) \((m = 3, 4, \ldots)\), the graphs \((C_m \lor T_1) \cup (C_m \circ K_1)\) and \((C_m \lor T_2) \cup (C_m \circ K_1)\) both are of order \(9 + 2m\) \((m = 3, 4, \ldots)\). Now, Corollary 23 and Lemma 3 implies that these two graphs are equienergetic. ■
In this section we just give the spectrum of the entries in the first row are the eigenvalues with multiplicity written below, and, respectively. Then the adjacency spectrum of \( G = (G_1 \lor G_2) \lor (G_2 \lor G_3) \lor (G_1 \lor G_4) \) is

\[
\sigma(G) = \begin{pmatrix}
\gamma_i & r_3 & \eta_j & r_4 \\
\frac{r_2 n}{2} & (r_2 - 2)n/2 & \frac{r_3 n}{2} & (r_1 - 2)n/2 \\
1 & \frac{\lambda_4 + \sqrt{4p(\lambda_k + r_4) + (\lambda_k - r_4)^2}}{2} & \frac{\mu_4 + \sqrt{4l(\mu_s + r_4) + (\mu_s - r_4)^2}}{2} & x_i \\
1 & 1 & 1 & 1
\end{pmatrix},
\]

where \( i = 2 \) to \( l \), \( j = 2 \) to \( p \), \( k = 2 \) to \( n \), \( s = 2 \) to \( m \), \( t = 1, 2, 3, 4 \). Also, the entries in the first row are the eigenvalues with multiplicity written below, and \( x_i \)'s are the roots of the polynomial

\[
((x - r_1)(x - r_4) - 2pr_4)((x - r_2)(x - r_3) - 2r_2l) - nm(x - r_3)(x - r_4).
\]

6. Spectra of \( (G_1 \lor G_2) \lor (G_2 \lor G_3) \lor (G_1 \lor G_4) \)

In this section we just give the spectrum of \( (G_1 \lor G_2) \lor (G_2 \lor G_3) \lor (G_1 \lor G_4) \), \( (G_1 \lor G_2) \lor (G_2 \lor G_3) \lor (G_1 \lor G_4) \) and \( (G_1 \lor G_2) \lor (G_2 \lor G_3) \lor (G_1 \lor G_4) \), where \( G_1, G_2, G_3 \) and \( G_4 \) are regular graphs on \( n, m, l \) and \( p \) vertices, respectively. Proofs of Theorems 26–28 are similar to the proof of Theorem 4.

**Theorem 26.** Let \( G_i \) be \( r_i \)-regular graphs \( (i = 1, 2, 3, 4) \), \( r_1 \geq 2 \) and \( r_2 \geq 2 \). Suppose \( \sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n), \sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_m), \sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l) \) and \( \sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p) \) are the adjacency spectrum of \( G_1, G_2, G_3 \) and \( G_4 \), respectively. Then the adjacency spectrum of \( G = (G_1 \lor G_2) \lor (G_2 \lor G_3) \lor (G_1 \lor G_4) \) is

\[
\sigma(G) = \begin{pmatrix}
\gamma_i & \eta_j & \sqrt{4p(\lambda_k + r_4) + (\lambda_k - r_4)^2}/2 \\
\sqrt{4l(\mu_s + r_4) + (\mu_s - r_4)^2}/2 & x_i \\
1 & 1 & 1 & 1
\end{pmatrix},
\]

where \( i = 2 \) to \( l \), \( j = 2 \) to \( p \), \( k = 2 \) to \( n \), \( s = 2 \) to \( m \), \( t = 1, 2, 3, 4 \). Also, the entries in the first row are the eigenvalues with multiplicity written below, and \( x_i \)'s are the roots of the polynomial

\[
((x - r_2)(x - r_3) - l)((x - r_1)(x - r_4) - pr_4) - nm(x - r_3)(x - r_4).
\]
\textbf{Theorem 27.} Let $G_i$ be $r_i$-regular graphs $(i = 1, 2, 3, 4)$ and $r_i \geq 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_n)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of $G_1$, $G_2$, $G_3$ and $G_4$, respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_2 \lor G_3) \cup (G_1 \lor G_4)$ is
\[
\sigma(G) = \begin{pmatrix}
\gamma_i & \eta_j & r_4 \\
n & m & (r_1 - 2)n/2
\end{pmatrix}
\begin{pmatrix}
\lambda_k + r_4 \pm \sqrt{4p(\lambda_k + r_1) + (\lambda_k - r_4)^2} / 2 \\
1
\end{pmatrix}
\begin{pmatrix}
\mu_s + r_3 \pm \sqrt{4l + (\mu_s - r_3)^2} / 2 \\
1
\end{pmatrix}
\begin{pmatrix}
x_t \\
1
\end{pmatrix},
\]
where $i = 2$ to $l$, $j = 2$ to $p$, $k = 2$ to $n$, $s = 2$ to $m$, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_t$'s are the roots of the polynomial
\[(x - r_1)(x - r_4) - 2pr_1)(x - r_2)(x - r_3) - l nm(x - r_3)(x - r_4).
\]

\textbf{Theorem 28.} Let $G_i$ be $r_i$-regular graphs $(i = 1, 2, 3, 4)$ and $r_i \geq 2$. Suppose $\sigma(G_1) = (\lambda_1 = r_1, \lambda_2, \ldots, \lambda_n)$, $\sigma(G_2) = (\mu_1 = r_2, \mu_2, \ldots, \mu_n)$, $\sigma(G_3) = (\gamma_1 = r_3, \gamma_2, \ldots, \gamma_l)$ and $\sigma(G_4) = (\eta_1 = r_4, \eta_2, \ldots, \eta_p)$ are the adjacency spectrum of $G_1$, $G_2$, $G_3$ and $G_4$, respectively. Then the adjacency spectrum of $G = (G_1 \lor G_2) \cup (G_2 \lor G_3) \cup (G_1 \lor G_4)$ is
\[
\sigma(G) = \begin{pmatrix}
\gamma_i & \eta_j & r_4 \\
n & m & (r_1 - 2)n/2
\end{pmatrix}
\begin{pmatrix}
\lambda_k + r_4 \pm \sqrt{4p(\lambda_k + r_1) + (\lambda_k - r_4)^2} / 2 \\
1
\end{pmatrix}
\begin{pmatrix}
\mu_s + r_3 \pm \sqrt{4l + (\mu_s - r_3)^2} / 2 \\
1
\end{pmatrix}
\begin{pmatrix}
x_t \\
1
\end{pmatrix},
\]
where $i = 2$ to $l$, $j = 2$ to $p$, $k = 2$ to $n$, $s = 2$ to $m$, $t = 1, 2, 3, 4$. Also, the entries in the first row are the eigenvalues with multiplicity written below, and $x_t$'s are the roots of the polynomial
\[(x - r_1)(x - r_4) - 2pr_1)(x - r_2)(x - r_3) - l nm(x - r_3)(x - r_4).
\]

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