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Research Article

Leap Eccentric Connectivity Index of Subdivision Graphs

Ali Ghalavand,[1] **Shiladhar Pawar**,[2] **and Nandappa D. Soner**[2]

[1]*Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran*
[2]*Department of Studies in Mathematics, University of Mysore, Manasagangotri 570006, Mysuru, India*

Correspondence should be addressed to Ali Ghalavand; alighalavand@grad.kashanu.ac.ir

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The second degree of a vertex in a simple graph is defined as the number of its second neighbors. The leap eccentric connectivity index of a graph M, $L\xi^c(M)$, is the sum of the product of the second degree and the eccentricity of every vertex in M. In this paper, some lower and upper bounds of $L\xi^c(S(M))$ in terms of the numbers of vertices and edges, diameter, and the first Zagreb and third leap Zagreb indices are obtained. Also, the exact values of $L\xi^c(S(M))$ for some well-known graphs are computed.

1. Introduction

In this paper, M is a finite and undirected simple graph. Let $V(M)$ and $E(M)$ be sets of vertices and edges of M, respectively. Then, we put $n = |V(M)|$ and $m = |E(G)|$. If $\{a, b\} \subseteq V(M)$, then the length of a shortest path connecting a and b in M is the distance between a and b in M and denoted by $d_M(a, b)$. Let x be a vertex of M, and let r be a positive integer. Then, the open r-neighborhood of x in M, $N_r(x)$, is the set of all vertices at distance r from x; that is, $N_r(x) = \{v \in V(M): d_M(v, x) = r\}$. The r-distance degree of a vertex x in M is the size of the open r-neighborhood of x in M, and it is denoted by $d_r(x/M)$ or simply $d_r(x)$ if no misunderstanding is possible; that is, $d_r(x/M) = d_r(x) = |N_r(x)|$. It is clear that $d_1(x/M)$ is the degree of vertex x in M, and we denoted it by $d_M(x)$ or simply $d(x)$. Also, the eccentricity of a vertex x in M, $e(v)$, is defined as $e(v) = \max\{d_M(v, u): u \in V(M)\}$, and the diameter and radius of graph M are defined as $diam(M) = \max\{e(v): v \in V(M)\}$ and $\mathrm{rad}(M) = \min\{e(v): v \in V(M)\}$, respectively.

The subdivision graph $S(M)$ of a simple graph M is the graph obtained from M by inserting an additional vertex into each edge of M, or equivalently, by replacing each of its edges with a path of length 2 [1].

The wheel graph $W_{1,q}$ of order $q+1$ is the join of K_1 and C_q in which K_1 is the complete graph with one vertex, and C_q is the q -vertex cycle graph. Clearly, $|V(W_{1,q})| = q+1$ and $|E(W_{1,q})| = 2q$. The apex vertex of the wheel is the vertex corresponding to K_1, and the rim vertices of the wheel are the vertices corresponding to C_q [2]. Note that all notions and notations not defined here can be obtained from the book of Harary [2].

In chemical graph theory, a numerical parameter of a given graph that is applicable in some chemical problems is called a topological index. The Zagreb group indices are two degree-based topological indices that were defined by Gutman and Trinajestic [3] in 1972 and elaborated in [4]. These indices are defined as

$$M_1(M) = \sum_{x\in V(M)} d_M(x)^2,$$
$$M_2(M) = \sum_{ab\in E(M)} d_M(a)d_M(b). \tag{1}$$

For the main properties of these two indices, we refer the interested readers to [3–7].

In 2017, Naji et al. [8] introduced three topological indices depending on the second degree of vertices. These invariants are so-called leap Zagreb topological indices and can be defined as follows:

$$\begin{aligned}\mathrm{LM}_1(M) &= \sum_{v\in V(M)} d_2(v)^2,\\ \mathrm{LM}_2(M) &= \sum_{uv\in E(M)} d_2(u)d_2(v),\\ \mathrm{LM}_3(M) &= \sum_{v\in V(M)} d(u)d_2(v).\end{aligned} \tag{2}$$

In [9], the first leap Zagreb topological index of some graph operations is computed, and in [10], some formulas for the leap Zagreb indices of generalized *rts* point line transformation graphs $T^{rts}(M)$, when $s = 1$, are obtained. We refer to [8–14] for more details on the leap Zagreb indices of graphs. In [15], Sharma et al. introduced the eccentric connectivity index of the graph M as $\xi^c(M) = \sum_{v\in V(M)} d(v)e(v)$. For mathematical properties, the interested readers can consult [15–17].

Recently, authors found in [18] introduced the leap eccentric connectivity index of a graph M. It is denoted by $L\xi^c(M)$ and can be defined $L\xi^c(M) = \sum_{v\in V(M)} d_2(v)e(v)$. They obtained the exact values of the leap eccentric connectivity index of complete, complete bipartite, cycle, path, and wheel graphs and determined some upper and lower bounds for $L\xi^c(M)$ in terms of the number of vertices, number of edges, diameter, total eccentricity, and Zagreb indices. In [19], the explicit formulas of the leap eccentric connectivity index for the Cartesian product, composition, disjunctions, symmetric difference, and corona product were computed.

In [20], exact values of $L\xi^c$ for thorny complete graphs, thorny complete bipartite graphs, thorny cycles, and thorny paths were reported. The authors of this paper also discussed some applications of the leap eccentric connectivity index of chemical structures such as cyclo-alkanes. In [21], some new upper and lower bounds for $L\xi^c(M)$ in the terms of the order, size, diameter, radius, and total eccentricity, Zagreb, and leap Zagreb indices are found. In the mentioned paper, some lower and upper bounds of $L\xi^c(S(M))$ in terms of the numbers of vertices and edges, diameter, and the first Zagreb and third leap Zagreb indices are also obtained. They also found the exact values of $L\xi^c(S(M))$ for some well-known graphs.

The following results of [18,22] are crucial in our arguments:

Theorem 1 (see [18]). *Let $n \geq 3$ be an integer. Then,*

$$L\xi^c(P_n) = \begin{cases} \dfrac{3n^2 - 10n + 12}{2}, & 2|n,\\[2ex] \dfrac{3n^2 - 10n + 11}{2}, & 2|n. \end{cases} \tag{3}$$

Theorem 2 (see [18]). *Let $n \geq 3$ be an integer. Then,*

$$L\xi^c(C_n) = \begin{cases} 0, & n = 3,\\ 8, & n = 4,\\ n^2, & n \neq 4, 2 \mid n,\\ n(n-1), & n \neq 3, 2\nmid n. \end{cases} \tag{4}$$

Lemma 1 (see [22]). *Let M be an n-vertex connected graph of size m. Then,*

$$d_2(v) \leq \left(\sum_{u\in N_1(v)} d_1(u)\right) - d_1(v). \tag{5}$$

The equality is attained if and only if G is a $\{C_3, C_4\}$-free graph.

By Lemma 1, for a (C_3, C_4)-free graph M, we have $\sum_{v\in V(G)} d_2(v) = M_1 - 2m$.

2. Main Results

The aim of this paper is to present the exact values of leap eccentric connectivity index of subdivision graph of some standard graphs.

Theorem 3. *Suppose $n \geq 3$. Then,*

$$L\xi^c(S(K_n)) = \begin{cases} 36, & \text{if } n = 3,\\ n(n-1)(4n-5), & \text{otherwise.} \end{cases} \tag{6}$$

Proof. Let $a_1, a_2, \ldots, a_n$ be the vertices of K_n, and let $b_1, b_2, \ldots, b_m$ be the new vertices added to K_n to obtain $S(K_n)$, where m is the size of K_n. Then, $d_2(a_i) = n - 1$, $d_2(b_j) = 2n - 4$, $e(a_i) = 3$, and $e(b_j) = \begin{cases} 3, & \text{if } n = 3,\\ 4, & \text{otherwise.}\end{cases}$

By definition, we have two following cases: □

Case 1. If $n = 3$, then

$$L\xi^c(S(K_3)) = \sum_{i=1}^{6} (2)(3) = 6(6) = 36. \tag{7}$$

Case 2. If $n \geq 4$, then

$$\begin{aligned} L\xi^c(S(K_n)) &= \sum_{w\in V(S(K_n))} d_2(w)e(w)\\ &= \sum_{i=1}^{n} d_2(a_i)e(a_i) + \sum_{i=1}^{m} d_2(b_j)e(b_j)\\ &= \sum_{i=1}^{n} (n-1)(3) + \sum_{j=1}^{m} (2n-4)(4)\\ &= 3n(n-1) + 8m(n-2). \end{aligned} \tag{8}$$

Since for the complete graph K_n, $m = n(n-1)/2$, it follows that

$$L\xi^c(S(K_n)) = n(n-1)(4n-5).$$

Theorem 4. *For $r \geq s \geq 2$, let $K_{r,s}$ be the complete bipartite graph. Then,*

$$L\xi^c\left(S\left(K_{r,s}\right)\right) = rs(3r + 3s + 2). \tag{9}$$

Proof. Suppose $r \geq s \geq 2$ and (V_1, V_2) is a partition of the vertex set, where $V_1 = \{v_1, v_2, v_3, \ldots, v_r\}, V_2 = \{u_1, u_2, u_3, \ldots, u_s\}$ and let $W = \{w_1, w_2, w_3, \ldots, w_{rs}\}$ be the set of new vertices in $S(K_{r,s})$. Then, $d_2(v_i) = s$, $d_2(u_j) = r$, $d_2(w_k) = r + s - 2$, $e(v_i) = 4$, $e(u_j) = 4$, and $e(w_k) = 3$. By definition,

$$\begin{aligned} L\xi^c\left(S\left(K_{r,s}\right)\right) &= \sum_{v_i \in V_1} d_2(v).e(v) + \sum_{u_j \in V_2} d_2(u).e(u) \\ &\quad + \sum_{w_k \in V_3} d_2(w).e(w) \\ &= \sum_{i=1}^{r} (s)(4) + \sum_{j=1}^{s} (r)(4) + \sum_{k=1}^{rs} (r + s - 2)(3) \\ &= 4rs + 4rs + 3rs(r + s - 2) \\ &= rs(3r + 3s + 2). \end{aligned} \tag{10}$$

□

Theorem 5. *Let $K_{1,n-1}$ be the star graph of order $n \geq 3$. Then,*

$$L\xi^c\left(S\left(K_{1,n-1}\right)\right) = 3n(n-1). \tag{11}$$

Proof. Let $v_0 \in K_{1,n-1}$, with $d(v_0) = n-1$, are be the central vertex, $v_1, v_2, \ldots, v_{n-1}$ are be the pendent vertices of $K_{1,n-1}$, and $u_1, u_2, \ldots, u_{n-1}$ are be the new vertices added to $K_{1,n-1}$, to obtain $S(K_{1,n-1})$. If $i = 1, 2, \ldots, n-1$, then $d_2(v_0) = n-1$, $d_2(v_i) = 1$, $d_2(u_i) = n-2$, $e(v_0) = 2$, $e(v_i) = 4$, and $e(u_i) = 3$. By definition,

$$\begin{aligned} L\xi^c\left(S\left(K_{1,n-1}\right)\right) &= d_2(v_0)e(v_0) + \sum_{i=1}^{n-1} d_2(v_i)e(v_i) + \sum_{j=1}^{n-1} d_2\left(u_j\right)e\left(u_j\right) \\ &= (n-1)(2) + \sum_{i=1}^{n-1} (1)(4) + \sum_{j=1}^{n-1} (n-2)(3) = 3n(n-1). \end{aligned} \tag{12}$$

□

Theorem 6. *Let $r \geq 1$ and $s \geq 1$ be two integers such that $n = r + s \geq 3$. Then,*

$$3n(n-1) \leq L\xi^c\left(S\left(K_{r,s}\right)\right) \leq \frac{1}{4}n^2(3n + 2). \tag{13}$$

On the left hand side, equality occurs if and only if $K_{r,s} \cong K_{1,n-1}$. On the right hand side, equality occurs if and only if $K_{r,s} \cong K_{(n/2),(n/2)}$.

Proof. We consider two cases as follows:

(i) $r = 1$ or $s = 1$. In this case, $G \cong K_{1,n-1}$, and by Theorem 5, $L\xi^c(S(K_{r,s})) = 3n(n-1)$.

(ii) $s, r \geq 2$. In this case, since $r + s = n$, $2(n-2) \leq rs \leq (n/2)(n/2)$. Now, by Theorem 4, $2(n-2)(3n+2) \leq L\xi^c(S(K_{r,s})) \leq (1/4)n^2(3n+2)$. On the left hand side, equality holds if and only if $K_{r,s} \cong K_{2,n-2}$. On the right hand side, equality holds if and only if $K_{r,s} \cong K_{n/2,n/2}$.

On the other hand, $2(n-2)(3n+2) - 3n(n-1) = n(3n-5) - 8 > 0$. Therefore, by (i) and (ii), $3n(n-1) \leq L\xi^c(S(K_{r,s})) \leq 1/4n^2(3n+2)$. On the left hand side, equality occurs if and only if $K_{r,s} \cong K_{1,n-1}$, and on the right hand side, equality occurs if and only if $K_{r,s} \cong K_{n/2,n/2}$. □

Proposition 1. *Let n be an integer. Then,*

$$L\xi^c(S(C_n)) = 4n^2. \tag{14}$$

Proof. Since $S(C_n) = C_{2n}$, the proof follows from Theorem 2. □

Proposition 2. *Let $n \geq 2$ be an integer. Then, $L\xi^c(S(P_n)) = 2(3n^2 - 8n + 6)$.*

Proof. Since $S(P_n) = P_{2n-1}$, the proof follows from Theorem 1. □

Theorem 7. *For* $n \geq 6$, $L\xi^c(S(W_{1,n})) = 2n(2n+23)$.

Proof. Let v_0 be the central vertex of $W_{1,n}$, $v_1, v_2, \ldots, v_n$, be the rim vertices $W_{1,n}$ and let $S(W_{1,n})$ be the subdivision of $W_{1,n}$. If w_i subdivides v_0v_i, $1 \leq i \leq n$, u_j subdivides v_jv_{j+1}, $1 \leq j \leq n-1$ and u_n subdivides v_nv_1. One can easily verify $L\xi^c(S(W_{1,n})) = 120$, if $n = 3$, $L\xi^c(S(W_{1,n})) = 204$, if $n = 4$ and $L\xi^c(S(W_{1,n})) = 310$, if $n = 5$. Let $n \geq 6$. Then, $d_2(v_0) = n$, $d_2(v_i) = 3$, $d_2(u_j) = 4$, $d_2(w_k) = n+1$, $e(v_0) = 3$, $e(v_i) = 5$, $e(u_j) = 6$, $e(w_k) = 4$, $1 \leq i, j, k \leq n$. By definition, we have

$$\begin{aligned} L\xi^c\left(S\left(W_{1,n}\right)\right) &= \sum_{v \in V\left(S\left(W_{1,n}\right)\right)} d_2(v)e(v) \\ &= d_2(v_0)e(v_0) + \sum_{i=1}^{n} d_2(v_i)e(v_i) + \sum_{i=1}^{n} d_2(u_j)e(u_j) + \sum_{i=1}^{n} d_2(w_k)e(w_k) \\ &= (n)(3) + \sum_{i=1}^{n}(3)(5) + \sum_{i=1}^{n}(4)(6) + \sum_{i=1}^{n}(n+1)(4) = 2n(2n+23). \end{aligned} \tag{15}$$

□

Theorem 8. *For natural numbers r and s, let* $D_{r,s}$ *be a double star with* $v_1, v_2, v_3, \ldots, v_r$ *be the pendent vertices have support at* v_0 *and* $u_1, u_2, u_3, \ldots, u_s$, *be the pendent vertices have support at* u_0. *Then,*

$$L\xi^c\left(S\left(D_{r,s}\right)\right) = 5\left(r^2 + s^2\right) + 13(r+s) + 8. \tag{16}$$

Proof. Let x_i subdivides v_0v_i, $1 \leq i \leq r$, y_j subdivides u_0u_i, $1 \leq j \leq s$, and w_0 subdivides v_0u_0. Then, $d_2(v_0) = r+1$, $d_2(u_0) = s+1$, $d_2(w_0) = r+s$, $d_2(v_i) = 1$, $d_2(u_j) = 1$, $d_2(x_i) = r$, $d_2(y_j) = s$, $e(v_0) = 4$, $e(u_0) = 4$, $e(w_0) = 3e(v_i) = 6$, $e(u_j) = 6$, $e(x_i) = 5$, and $e(y_j) = 5$. By definition, we have

$$\begin{aligned} L\xi^c\left(S\left(D_{r,s}\right)\right) &= d_2(v_0)e(v_0) + \sum_{i=1}^{r} d_2(x_i)e(x_i) + \sum_{i=1}^{r} d_2(v_i)e(v_i) + d_2(w_0)e(w_0) \\ &\quad + d_2(u_0)e(u_0) + \sum_{j=1}^{s} d_2(y_j)e(y_j) + \sum_{j=1}^{s} d_2(u_j)e(u_j) \\ &= (r+1)(4) + \sum_{i=1}^{r}(r)(5) + \sum_{i=1}^{r}(1)(6) + (r+s)(3) + (s+1)(4) \\ &\quad + \sum_{j=1}^{s}(s)(5) + \sum_{j=1}^{s}(1)(6) \\ &= 5\left(r^2 + s^2\right) + 13(r+s) + 8. \end{aligned} \tag{17}$$

□

Theorem 9. *Let* $n \geq 7$ *be a natural number. Then,*

$$L\xi^c\left(S\left(D_{i,n-2-i}\right)\right) > L\xi^c\left(S\left(D_{i+1,n-3-i}\right)\right) \text{ for } i = 1, 2, \ldots \left\lfloor \frac{n-2}{2} \right\rfloor - 1. \tag{18}$$

Proof. By Theorem 8,

$$L\xi^c\left(S\left(D_{i,n-2-i}\right)\right) - L\xi^c\left(S\left(D_{i+1,n-3-i}\right)\right) = 10(n-2i-3). \tag{19}$$

Now, if $2|n-2$, then by (19),

$$L\xi^c\left(S\left(D_{i,n-2-i}\right)\right) - L\xi^c\left(S\left(D_{i+1,n-3-i}\right)\right) \geq 10\left(n - 2\left(\frac{n-2}{2} - 1\right) - 3\right) = 10. \tag{20}$$

And if $2\nmid n-2$, then by (19),

$$L\xi^c\left(S\left(D_{i,n-2-i}\right)\right) - L\xi^c\left(S\left(D_{i+1,n-3-i}\right)\right) \geq 10\left(n - 2\left(\frac{n-3}{2} - 1\right) - 3\right) = 20. \tag{21}$$

Therefore, $L\xi^c(S(D_{i,n-2-i})) > L\xi^c(S(D_{i+1,n-3-i}))$ for $i = 1, 2, \ldots \lfloor n-2/2 \rfloor - 1$. □

Corollary 1. *Let r, s, and n be three natural numbers such that $r + s + 2 = n \geq 7$. Then,*

$$\frac{5}{2}n^2 + 3n - 8 \leq L\xi^c\left(S\left(D_{r,s}\right)\right) \leq 5n^2 - 17n + 32, \quad 2 \mid n-2,$$

$$\frac{1}{2}(5n+1)(n-3) \leq L\xi^c\left(S\left(D_{r,s}\right)\right) \leq 5n^2 - 17n + 32, \quad 2 \mid n-2. \tag{22}$$

On the left hand side, equalities occur if and only if $D_{r,s} \cong D_{\lfloor n-2/2 \rfloor, \lfloor n-2/2 \rfloor}$. On the right hand side, equalities occur if and only if $D_{r,s} \cong D_{1,n-3}$.

Theorem 10. *Let M be an n-vertex connected graph of size m such that $n \geq 3$. Then,*

$$L\xi^c(M) \leq nM_1(M) - 2nm - LM_3(M). \tag{23}$$

The bound is attained for P_4.

Proof. Since $e(v) \leq n - d(v)$ for every $v \in V(M)$,

$$\begin{aligned} L\xi^c(M) &= \sum_{v \in V(M)} d_2(v) e(v) \leq \sum_{v \in V(M)} d_2(v)(n - d(v)) \\ &= \sum_{v \in V(M)} n d_2(v) - \sum d_1(v) d_2(v) \\ &= n \sum_{v \in V(M)} d_2(v) - \sum d_1(v) d_2(v). \end{aligned} \tag{24}$$

Using definition of $LM_3(M)$ and Lemma 3, we get

$$\begin{aligned} L\xi^c(M) &\leq n \sum_{v \in V(M)} \left(\sum_{uv \in E(M)} d(u) - d(v) \right) - \mathrm{LM}_3(M) \\ &= n \sum_{v \in V(M)} d(v)^2 - 2nm - \mathrm{LM}_3(M) \\ &= nM_1(M) - 2nm - \mathrm{LM}_3(M). \end{aligned} \tag{25}$$

□

Corollary 2. *Let M be an n-vertex connected graph of size m such that $n \geq 3$. Then,*

$$L\xi^c(S(M)) \leq (n + m - 3)M_1(M) + 4m. \tag{26}$$

Proof. For $uv \in E(M)$, let v_{uv} be the new vertex of degree 2 on uv in $S(M)$. By definition of $S(M)$, $d(v/S(M)) = d(v/M)$, $d_2(v/S(M)) = d_2(v/M)$ for $v \in V(M)$ and $d(v_{uv}/S(M)) = 2$, $d_2(v_{uv}/S(M)) = d(u/M) + d(v/M) - 2$ for $uv \in E(M)$. Therefore, $M_1(S(M)) = M_1(M) + 4m$ and $LM_3(S(M)) = M_1(M) + 2M_1(M) - 4m = 3M_1(M) - 4m$. So, by Theorem 10, $L\xi^c(M) \leq (n + m - 3)M_1(M) + 4m$. □

Theorem 11. *Let M be an n-vertex connected graph of size $m \geq 2$. Then, $L\xi^c(S(G)) \geq 2(n+m)$.*

Proof. Let $V_0 = \{v \in V(M); d(v) = n-1\}$ and $n_0 = |V_0|$. Then, $d_2(v) = 0$ for every $v \in V_0$ and for every $u \in V\ V_0$, we have $e(u) \geq 2$ and $d_2(u) \geq 1$. Hence,

$$\begin{aligned} L\xi^c(M) &= \sum_{v \in V_0} d_2(v) e(v) + \sum_{v \in V/V_0} d_2(v) e(v) \\ &\geq \sum_{v \in V_0} (0)(1) + \sum_{v \in V/V_0} (1)(2) \\ &= 2\left|V/V_0\right| \\ &= 2(n - n_0). \end{aligned} \tag{27}$$

Now, it is easy to see that the number of vertices of $S(M)$ is $n+m$, and the number of vertices of degree $n-1$ in $S(M)$ is zero. Therefore, by (27), we have $L\xi^c(S(M)) \geq 2(n(S(M)) - n_0(S(M)) = 2(n+m)$. □

Theorem 12. *Let M be an n-vertex graph of size m. Then,*

$$L\xi^c(M) \leq diam(M)(M_1(M) + 2m). \tag{28}$$

The equality occurs if and only if M is a self-centered and $\{C_3, C_4\}$-free graph.

Proof. By definition, for all $v \in V(M)$, $e(v) \leq diam(M)$, the equality holds if and only if M is a self-centered. Also, by

Lemma 3, $\sum_{v\in V(M)} d_2(v) \le M_1(M) - 2m$, and the equality occurs if and only if M is a $\{C_3, C_4\}$-free graph. Therefore,

$$\begin{aligned} L\xi^c(M) &= \sum_{v\in V(M)} d_2(v)e(v) \\ &\le \sum_{v\in V(M)} d_2(v) diam(M) \\ &\le diam(M)(M_1(M) - 2m). \end{aligned} \tag{29}$$

The equalities hold if and only if M is a self-centered and $\{C_3, C_4\}$-free graph. □

Corollary 3. *Let M be an n-vertex graph of size m. Then,*

$$L\xi^c(S(M)) \le diam(S(M))M_1(M). \tag{30}$$

The equality occurs if and only if $S(M)$ is a self-centered.

Theorem 13. *Let M be an n-vertex connected graph of size m such that $n \ge 4$. Then, $L\xi^c(S(M)) \ge 4M_1(M) - 2m$, the equality occurs if and only if $M \cong K_n$.*

Proof. By definition of $S(M)$, for all $v \in V(M)$, $d_2(v/S(M)) = d(v/M)$, $e(v) \ge 3$, and the equalities occur if and only if $M \cong K_n$. Also, for all $uv \in E(M)$, $d_2(v_{uv}/S(M)) = d(v/M) + d(u/M) - 2$, $e(v_{uv}) \ge 4$, and the equalities occur if and only if $M \cong K_n$. Therefore, by definitions of $L\xi^c$ and $S(M)$, we have

$$\begin{aligned} L\xi^c(S(M)) &= \sum_{v\in V(M)} d_2\left(\frac{v}{S(M)}\right)e\left(\frac{v}{S(M)}\right) + \sum_{uv\in E(M)} d_2\left(\frac{v_{uv}}{S(M)}\right)e\left(\frac{v_{uv}}{S(M)}\right) \\ &\ge \sum_{v\in V(M)} 3d\left(\frac{v}{M}\right) + \sum_{uv\in E(M)} 4\left(d\left(\frac{u}{M}\right) + d\left(\frac{v}{M}\right) - 2\right) \\ &= 6m + 4M_1(M) - 8m \\ &= 4M_1(M) - 2m. \end{aligned} \tag{31}$$

The equality occurs if and only if $M \cong K_n$. □

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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