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Crossed out modular equations of degree 11 of Ramanujan

K. R. Vasuki[1] · **M. V. Yathirajsharma**[2]

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Abstract
In this paper, we confirm two modular equations of degree 11 of S. Ramanujan which were crossed out by him on page 244 of his second notebook.

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1 Introduction

In Chapter 20 of his second notebook [2, pp. 243–244], Ramanujan recorded modular equations of degree 11. In fact, Ramanujan recorded nine modular equations of degree 11 and crossed out the last two of them. Berndt [1] proved all first seven modular equations. He employed the theory of theta functions to prove the first one in the list and proved the remaining six by employing the theory of modular forms. Regarding the last two modular equations, Berndt just remarked that “Ramanujan has crossed them out.” The purpose of this paper is to confirm those two crossed out modular equations of Ramanujan.

We complete this section by recalling some definitions and notations. Let a be a complex number and n be a positive integer. In what follows, we employ usual notation:

$$(a)_0 = 1$$

✉ K. R. Vasuki
vasuki_kr@hotmail.com

M. V. Yathirajsharma
yathirajsharma@gmail.com

1 Department of Studies in Mathematics, University of Mysore, Manasagangothri Campus, Mysuru 570006, India

2 Sarada Vilas College, Krishnamurthypuram, Mysuru 570004, India

and

$$(a)_n = a(a+1)\dots(a+n-1).$$

Gauss hypergeometric series ${}_2F_1(a, b; c; z)$ is defined by

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \quad |z| < 1. \tag{1.1}$$

Let

$$F(x) = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)},$$

with $0 < x < 1$. Suppose that $0 < \alpha, \beta < 1$ and that

$$F(\beta) = nF(\alpha)$$

holds for some positive integer n. Then any relation between α and β induced by the above is called a modular equation of degree n. In such cases, we say that β is of degree n over α. For $0 < \alpha < 1$, if we set

$$q = e^{-\pi F(\alpha)},$$

then $0 < q < 1$ and

$$q^n = e^{-\pi F(\beta)},$$

whenever β is of degree n over α. One can easily observe that, we may treat α and β as functions of q from $(0, 1) \to (0, 1)$. Further, we may extend α and β as continuous functions to $[0, 1]$, with $\alpha(0) = \beta(0) = 0$ and $\alpha(1) = \beta(1) = 1$.

2 Crossed out modular equations of degree 11

In this section, we confirm the two crossed out modular equations of degree 11 of Ramanujan. Before going to the proofs, we shall introduce the parameter 'a.' For this, we recall the first modular equation of degree 11 of Ramanujan [2, p. 243]:

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1. \tag{2.1}$$

For a proof of the above using the theory of theta functions, see [1, pp. 364–366]. We now define the parameter 'a' as follows:

$$a = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}. \tag{2.2}$$

Using the above in (2.1), we see that

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1 - 2a. \tag{2.3}$$

Squaring both sides of the above and using the definition of a, we obtain

$$(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} = 1 - 4a + 4a^2 - a^3. \tag{2.4}$$

Squaring both sides of (2.4) and using the definition of a, we find that

$$\alpha + \beta - 2\alpha\beta = \frac{a}{2}\left(16 - 48a + 68a^2 - 48a^3 + 16a^4 - a^5\right). \tag{2.5}$$

We now state the two crossed out modular equations as a theorem and supply the proof to it.

Theorem 2.1 *If β is of degree 11 over α, then the following hold:*

$$\begin{aligned} &2^{1/3}\left(\frac{\alpha^{11}(1-\beta)^{11}}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{\beta^{11}(1-\alpha)^{11}}{\alpha(1-\beta)}\right)^{1/24} \\ &\quad = \left(3 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right) \\ &\qquad \times \left(\frac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2}. \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &4 - 2^{1/3}\left(\frac{\alpha^{11}(1-\beta)^{11}}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{\beta^{11}(1-\alpha)^{11}}{\alpha(1-\beta)}\right)^{1/24} \\ &\quad = 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}\left\{2 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right\}. \end{aligned} \tag{2.7}$$

Proof We have

$$\begin{aligned} \left(\{\alpha(1-\beta)\}^{1/2} - \{\beta(1-\alpha)\}^{1/2}\right)^2 &= \alpha + \beta - 2\alpha\beta - \frac{a^6}{2} \\ &= a(2 - 4a + 4a^2 - a^3)(2-a)^2, \end{aligned} \tag{2.8}$$

where we have used (2.2) and (2.5). From (2.4), we observe that

$$2 - 4a + 4a^2 - a^3 = 1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} > 0.$$

Also from (2.3), it follows that

$$2 - a = \left(3 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right)/2 > 0.$$

Also $\alpha > \beta$ by the definition of modular equations. Hence

$$\alpha(1-\beta) > \beta(1-\alpha).$$

Hence, from (2.8), it follows that

$$\{\alpha(1-\beta)\}^{1/2} - \{\beta(1-\alpha)\}^{1/2} = \frac{\sqrt{a}}{\sqrt{2}}\left(\frac{2-4a+4a^2-a^3}{2}\right)^{1/2}(4-2a),$$

which in turn implies that

$$\frac{\sqrt{2}}{\sqrt{a}}\left(\{\alpha(1-\beta)\}^{1/2} - \{\beta(1-\alpha)\}^{1/2}\right) = \left(\frac{1+(1-4a+4a^2-a^3)}{2}\right)^{1/2} \times (3+(1-2a)). \tag{2.9}$$

Consider

$$2^{1/3}\left(\frac{\alpha^{11}(1-\beta)^{11}}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{\beta^{11}(1-\alpha)^{11}}{\alpha(1-\beta)}\right)^{1/24} = 2^{1/3}\left(\frac{\{\alpha(1-\beta)\}^{1/2} - \{\beta(1-\alpha)\}^{1/2}}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}}\right).$$

Using now the definition of a in the above, we obtain

$$2^{1/3}\left(\frac{\alpha^{11}(1-\beta)^{11}}{\beta(1-\alpha)}\right)^{1/24} - 2^{1/3}\left(\frac{\beta^{11}(1-\alpha)^{11}}{\alpha(1-\beta)}\right)^{1/24} = \frac{\sqrt{2}}{\sqrt{a}}\left(\{\alpha(1-\beta)\}^{1/2} - \{\beta(1-\alpha)\}^{1/2}\right). \tag{2.10}$$

Comparing (2.9) and (2.10) and then using equations (2.3) and (2.4) on the right-hand side of (2.9), we complete the proof of (2.6).

If we set

$$x = 2^{1/3}\left(\frac{\alpha^{11}(1-\beta)^{11}}{\beta(1-\alpha)}\right)^{1/24} \quad \text{and} \quad y = 2^{1/3}\left(\frac{\beta^{11}(1-\alpha)^{11}}{\alpha(1-\beta)}\right)^{1/24},$$

then the left-hand side of (2.7) can be written as $4-(x+y)$. We can also write (2.6) as

$$x - y = (4-2a)\left(\frac{2-4a+4a^2-a^3}{2}\right)^{1/2},$$

where we have used (2.4) and(2.3). We now have

$$x + y = \sqrt{(x-y)^2+4xy} = 2(2a^2-3a+2).$$

We have used the following argument to decide the sign. Treating α and β as continuous functions of q on [0, 1], if we define $f(q) = 2a^2 - 3a + 2$ (where a is a function of q), then $f(0) = f(1) = 2 > 0$. If $f(q)$ assumes any negative value for $q \in (0, 1)$, then $f(q)$ must have a root in (0, 1). If that is so, then for some $q \in (0, 1)$, we have

$$2a^2 - 3a + 2 = 0.$$

This implies,

$$a = \frac{3 \pm \sqrt{-7}}{4},$$

which is simply absurd as a is a real valued function of q on (0, 1). Hence $f(q)$ can never be zero. In other words,

$$2a^2 - 3a + 2 > 0$$

always whenever $q \in (0, 1)$. Observing that $4 - (x + y) = 2a(3 - 2a)$ and on using (2.2) and (2.3), we obtain (2.7). This completes the proof. □

Remark The modular equations (2.6) and (2.7) stated in Theorem 2.1 are equivalent to the following two theta function identities, respectively:

$$\begin{aligned} &4\chi(q)\chi(q^{11})\left\{\psi^2(q^2)\varphi^2(-q^{11}) + q^5\psi^2(q^{22})\varphi^2(-q)\right\} \\ &= \left\{3\varphi(q)\varphi(q^{11}) + 4q^3\psi(q^2)\psi(q^{22}) + \varphi(-q)\varphi(-q^{11})\right\} \\ &\times \left\{\varphi(q^2)\varphi(q^{22}) + 4q^6\psi(q^4)\psi(q^{44})\right\} \end{aligned}$$

and

$$\begin{aligned} &\varphi^2(q)\varphi^2(q^{11}) - \chi(q)\chi(q^{11})\left\{\psi^2(q^2)\varphi^2(-q^{11}) + q^5\psi^2(q^{22})\varphi^2(-q)\right\} \\ &= qf(-q^2)f(-q^{22})\left\{2\varphi(q)\varphi(q^{11}) + 4q^3\psi(q^2)\psi(q^{22}) + \varphi(-q)\varphi(-q^{11})\right\}, \end{aligned}$$

where, for $|q| < 1$, we have

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \ \psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}, \ f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}},$$

and

$$\chi(q) = \prod_{n=0}^{\infty}((1 + q^{2n+1}).$$

One can use Entry 10, Entry 11, and Entry 12 of Chapter 17 [1, pp. 122–124] of the second notebook of Ramanujan to prove the equivalence of the above theta function identities and the modular equations (2.6) and (2.7).

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