Indian J Pure Appl Math (2022) 53:1110–1121
https://doi.org/10.1007/s13226-022-00222-z

ORIGINAL RESEARCH

Evaluation of convolution sums $\sum_{l+15m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+5m=n} \sigma(l)\sigma(m)$

K. Pushpa · **K. R. Vasuki**

Received: 10 September 2020 / Accepted: 14 January 2022 / Published online: 31 January 2022

Abstract In this article, we have evaluated the convolution sums $\sum_{al+bm=n} \sigma(l)\sigma(m)$ for $a \cdot b = 15$, where $a, b \in \mathbb{N}$, using an elementary method. The deduced convolution sums are in a little elegent form than that derived by B. Ramakrishnan and B. Sahu [24]. As a consequence, we determine a formula for the number of representations of a positive integer n by the octonary quadratic form

$$\left(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\right) + 5\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right).$$

Keywords Dedekind eta function · Ramanujan's theta function · Eisenstein series · Divisors function

Mathematics subject classification 11A25 · 11E20 · 11E25 · 11F20 · 11M36

1 Introduction

Let $\mathbb{N}$ denote the set of all natural numbers. As usual let

$$\sigma_k(n) := \sum_{d|n} d^k, \qquad n, d, k \in \mathbb{N}$$

and

$$\sigma_k(n) = 0 \quad \forall \quad n \notin \mathbb{N}.$$

For convenience, we set $\sigma_1(n) = \sigma(n)$. We define the convolution sum $W_k(n)$ by

$$W_k(n) := \sum_{m<\frac{n}{k}} \sigma(m)\sigma(n-km)$$

Communicated by B. Sury.

K. Pushpa (✉)· K. R. Vasuki
Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru 570 006, India
E-mail: pushpamys94@gmail.com

K. R. Vasuki
E-mail: vasuki_kr@hotmail.com

and

$$W_{(a,b)}(n) := \sum_{\substack{l,m \\ al+bm=n}} \sigma(l)\sigma(m),$$

where $a, b, k, n \in \mathbb{N}$. Note that $W_{(1,k)}(n) = W_{(k,1)}(n) = W_k(n)$. M. Besge in his paper [11], obtained the formula

$$W_1 = \frac{5}{12}\sigma_3(n) + \frac{1-6n}{12}\sigma(n). \tag{1.1}$$

The above found to be the first work in the evaluation of convolution sum for divisor function. J. W. L. Glaisher [16], [17] and S. Ramanujan [29] have also deduced (1.1). Below is the listed table on the works of convolution sum $W_k(n)$and $W_{(a,b)}(n)$ inspired by the above works.

k and (a, b)	Authors	References
1	M. Besge, J. W. L. Glaisher, S. Ramanujan	[11,16,29]
2, 3, 4	J. G. Huard et.al	[18]
5, 7	M. Lemire & K. S. Williams, S. Cooper & P. C. Toh	[14,19]
6, (2,3)	Ş. Alaca & K. S. Williams	[6]
8, 9	K. S. Williams	[32] [31]
10, 11 , 13, 14	E. Royer	[30]
(2,5), (4,5), 20	S. Cooper and D. Ye	[15]
12, (3,4), 16, 18, (2,9), 24, (3,8)	A. Alaca & Ş. Alaca & K. S. Williams	[1–4]
12, (3,4), 15, (3,5)	B. Ramakrishnan & B. Sahu	[24,25]
7, (2,7), 14, (3,7), 21, (4,7), 28	B. Ramakrishnan & B. Sahu & A. K. Singh	[26]
23	H. H. Chan & S. Cooper	[12]
25	E. X. W. Xia & X. L. Tian & O. X. M. Yao.	[33]
27, 32	Ş. Alaca & Y. Kesicioğlu	[5]
36,(4,9)	D. Ye	[34]
14, (2,7), 26, (2,13), 28, (4,7), 30, (2,15), (3,10), (5,6), 22, (2,11), 44, (4,11), 52, (4,13)	E. Ntienjem.	[21] [22,23]

The main objective of this paper is to evaluate the convolution sums for level 15 i.e., W_{15} and $W_{(3,5)}$ which has already been done by B. Ramakrishnan and B. Sahu [24]. Our formula for level 15 is little elegant than that in [24]. We use an elementary method to evaluate these sums. As an application, we determine a formula for the number of representations of a positive integer n by the octonary quadratic form

$$\left(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\right) + 5\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right).$$

2 Preliminary results

In this section, we recall definitions and known results which are required to prove our main identities. For $q = e^{2\pi i\tau}$, $Im(\tau) > 0$, the Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau) = q^{\frac{1}{24}}(q; q)_\infty,$$

where

$$(q; q)_\infty = \prod_{n=1}^{\infty}(1 - q^n) = f(-q),$$

one of the Ramanujan's theta functions. For any positive integer n, we set

$$f_n = f(-q^n),\ A_n = \frac{f_n}{q^{\frac{n}{12}} f_{3n}},\ B_n = \frac{f_n}{q^{\frac{n}{6}} f_{5n}}.$$

Further for convenience throughout this paper, we set

$$u = \frac{1}{B_1 B_3},\ v = \frac{1}{A_1 A_5},\ w = \frac{B_1}{B_3},\ t = \frac{1}{w^3} - w^3.$$

We make use of the following interesting three theta function identities of Ramanujan found in the unorganized portion of his second note book [27].

Theorem 2.1 *[9, Entry 62, p.221], [27, p.324] We have*

$$(A_1 A_5)^2 + 5 + \frac{9}{(A_1 A_5)^2} = \left(\frac{A_5}{A_1}\right)^3 - \left(\frac{A_1}{A_5}\right)^3,$$

equivalently

$$\frac{1}{v^2} + 5 + 9v^2 = \frac{1}{w^3} - w^3.$$

Employing the definition of t in the right hand side of the above theorem, we have

$$(A_1 A_5)^6 + \frac{9^3}{(A_1 A_5)^6} = (t-5)\left(t^2 - 10t - 2\right) \tag{2.1}$$

and

$$(A_1 A_5)^6 - \frac{9^3}{(A_1 A_5)^6} = (t-2)(t-8)\sqrt{(t+1)(t-11)}. \tag{2.2}$$

Positive sign is considered in the above, since $A_1 A_5 > 1$, which follows from its series expansion,

$$A_1 A_5 = \frac{1}{q^{\frac{1}{2}}}\left(1 - q - q^2 + q^3 - q^4 + \cdots\right).$$

Again by the definition of t and w, we find that

$$\left(\frac{A_5}{A_1}\right)^6 + \left(\frac{A_1}{A_5}\right)^6 = t^2 + 2 \tag{2.3}$$

and

$$\left(\frac{A_5}{A_1}\right)^6 - \left(\frac{A_1}{A_5}\right)^6 = t\sqrt{t^2 + 4}. \tag{2.4}$$

Positive sign is considered in the above, since $\frac{A_5}{A_1} > 1$, which follows from its series expansion,

$$\frac{A_5}{A_1} = \frac{1}{q^{\frac{1}{3}}}\left(1 + q + 2q^2 + 2q^3 + 4q^4 + \cdots\right).$$

Multiplying (2.1) and (2.3), we obtain

$$A_1^{12} + \frac{9^3}{A_1^{12}} + A_5^{12} + \frac{9^3}{A_5^{12}} = (t-5)\left(t^2+2\right)\left(t^2 - 10t - 2\right).$$

Likewise multiplying (2.2) and (2.4), we obtain

$$-A_1^{12} - \frac{9^3}{A_1^{12}} + A_5^{12} + \frac{9^3}{A_5^{12}} = t(t-2)(t-8)\sqrt{(t+1)(t-11)\left(t^2+4\right)}.$$

From the above two identities, it is easy to find that

$$A_1^{12}+\frac{9^3}{A_1^{12}}=$$
$$\frac{1}{2}\left[(t-5)\left(t^2+2\right)\left(t^2-10t-2\right)-t\,(t-2)\,(t-8)\sqrt{(t+1)\,(t-11)\left(t^2+4\right)}\right] \tag{2.5}$$

and

$$A_5^{12}+\frac{9^3}{A_5^{12}}=$$
$$\frac{1}{2}\left[(t-5)\left(t^2+2\right)\left(t^2-10t-2\right)+t\,(t-2)\,(t-8)\sqrt{(t+1)\,(t-11)\left(t^2+4\right)}\right]. \tag{2.6}$$

Using the definition of t in the right hand side of equivalent form of the Theorem 2.1, we have

$$\frac{1}{v^2}+9v^2=t-5 \quad \text{and} \quad \frac{1}{v^2}-9v^2=\sqrt{(t+1)\,(t-11)}.$$

From the above equations, we obtain

$$\frac{1}{v^2}=\frac{(t-5)+\sqrt{(t+1)\,(t-11)}}{2} \tag{2.7}$$

and

$$v^2=\frac{(t-5)-\sqrt{(t+1)\,(t-11)}}{18}. \tag{2.8}$$

Theorem 2.2 *[9, Entry 63, p.223], [27, p.325] We have*

$$(B_1B_3)^3+\frac{125}{(B_1B_3)^3}=\left(\frac{B_3}{B_1}\right)^6-9\left(\frac{B_3}{B_1}\right)^3-9\left(\frac{B_1}{B_3}\right)^3-\left(\frac{B_1}{B_3}\right)^6,$$

equivalently

$$\frac{1}{u^3}+125u^3=\frac{1}{w^6}-\frac{9}{w^3}-9w^3-w^6. \tag{2.9}$$

Theorem 2.3 *[9, Entry 64, p.226], [27, p.323] We have*

$$(B_1B_3)^3-\frac{125}{(B_1B_3)^3}=\frac{1}{v^4}+\frac{1}{v^2}-9v^2-81v^4,$$

equivalently

$$\frac{1}{u^3}-125u^3=\frac{1}{v^4}+\frac{1}{v^2}-9v^2-81v^4.$$

Employing Theorem 2.1 in the right side of the above and from (2.9), we find that

$$\frac{1}{u^3}-125u^3=(t-4)\sqrt{(t+1)\,(t-11)} \quad \text{and} \quad \frac{1}{u^3}+125u^3=(t-9)\sqrt{\left(t^2+4\right)}.$$

From the above identities, we have

$$\frac{1}{u^3}=\frac{1}{2}\left[(t-9)\sqrt{\left(t^2+4\right)}+(t-4)\sqrt{(t+1)\,(t-11)}\right] \tag{2.10}$$

and

$$125u^3=\frac{1}{2}\left[(t-9)\sqrt{\left(t^2+4\right)}-(t-4)\sqrt{(t+1)\,(t-11)}\right]. \tag{2.11}$$

Now from (2.5) and (2.10), we deduce that

$$\frac{1}{u}\left(A_1^6+\frac{27}{A_1^6}\right)^{\frac{2}{3}}=3\sqrt{t^2+4}-2\sqrt{(t+1)\,(t-11)} \tag{2.12}$$

and from (2.6) and (2.11), we deduce that

$$5u\left(A_5^6+\frac{27}{A_5^6}\right)^{\frac{2}{3}}=3\sqrt{t^2+4}+2\sqrt{(t+1)\,(t-11)}. \tag{2.13}$$

Using the definition of t in Theorem 2.2 and Theorem 2.3, we find that

$$(B_1B_3)^3+\frac{125}{(B_1B_3)^3}=(t-9)\sqrt{t^2+4} \tag{2.14}$$

and

$$(B_1B_3)^3-\frac{125}{(B_1B_3)^3}=(t-4)\sqrt{(t+1)\,(t-11)}. \tag{2.15}$$

Again by the definition of t and w, we have

$$\left(\frac{B_3}{B_1}\right)^3+\left(\frac{B_1}{B_3}\right)^3=\sqrt{t^2+4} \tag{2.16}$$

and

$$\left(\frac{B_3}{B_1}\right)^3-\left(\frac{B_1}{B_3}\right)^3=t. \tag{2.17}$$

Multiplying (2.14) and (2.16), we obtain

$$B_1^6+\frac{125}{B_1^6}+B_3^6+\frac{125}{B_3^6}=(t-9)\left(t^2+4\right).$$

Likewise multiplying (2.15) and (2.17), we see that

$$-B_1^6-\frac{125}{B_1^6}+B_3^6+\frac{125}{B_3^6}=t\,(t-4)\sqrt{(t+1)\,(t-11)}.$$

Similarly multiplying (2.14) and (2.17) and multiplying (2.15) and (2.16), we have

$$-B_1^6+\frac{125}{B_1^6}+B_3^6-\frac{125}{B_3^6}=t\,(t-9)\sqrt{t^2+4}$$

and

$$B_1^6-\frac{125}{B_1^6}+B_3^6-\frac{125}{B_3^6}=(t-4)\sqrt{(t+1)\,(t-11)\left(t^2+4\right)}.$$

From the above four identities, we deduce the following identities:

$$\left(B_3^6+\frac{125}{B_3^6}+22\right)^{\frac{1}{2}}=\frac{(2t-1)+\sqrt{(t+1)\,(t-11)}}{3v}, \tag{2.18}$$

$$B_1^6-\frac{125}{B_1^6}=\frac{\sqrt{t^2+4}}{2}\left[(t-4)\sqrt{(t+1)\,(t-11)}-t\,(t-9)\right], \tag{2.19}$$

and

$$B_3^6-\frac{125}{B_3^6}=\frac{\sqrt{t^2+4}}{2}\left[(t-4)\sqrt{(t+1)(t-11)}+t(t-9)\right]. \tag{2.20}$$

Let $P(q)$ and $Q(q)$ denote the Eisenstein series of weight 2 and 4 respectively, defined by

$$P(q):=1-24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$$

and

$$Q(q):=1+240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}.$$

For any positive integer n, we set $P_n:=P(q^n)$ and $Q_n:=Q(q^n)$. We also require the following Eisenstein series identities:

Theorem 2.4 *We have*

$$P_n=1-24\sum_{k=1}^{\infty}\sigma(k)q^{kn} \tag{2.21}$$

$$Q_n=1+240\sum_{k=1}^{\infty}\sigma_3(k)q^{kn}, \tag{2.22}$$

and

$$(P(q))^2=1+\sum_{k=1}^{\infty}\left[240\sigma_3(k)-288k\sigma(k)\right]q^k. \tag{2.23}$$

For a proof of (2.21) and (2.22), see [7, p.318] and for a proof of (2.23), see Glaisher [16].

Theorem 2.5 *[27], [8, Chapter 21] We have*

$$-P_1+3P_3=2q^{\frac{1}{3}}f_1^2f_3^2\left(A_1^6+\frac{27}{A_1^6}\right)^{\frac{2}{3}} \tag{2.24}$$

and

$$-P_1+5P_5=4q^{\frac{1}{2}}f_1^2f_5^2\left(B_1^6+\frac{125}{B_1^6}+22\right)^{\frac{1}{2}}. \tag{2.25}$$

Theorem 2.6 *[13, p.228]We have*

$$(-P_1+3P_3)^2=\frac{4}{10}(Q_1+9Q_3). \tag{2.26}$$

In his lost notebook [28], Ramanujan recorded the following interesting Eisenstein series identities:

Theorem 2.7 *We have*

$$Q_1=qf_1^4f_5^4\left(B_1^6+\frac{3125}{B_1^6}+250\right) \tag{2.27}$$

and

$$Q_5=qf_1^4f_5^4\left(B_1^6+\frac{5}{B_1^6}+10\right). \tag{2.28}$$

For a proof of the above theorem, see B. C. Berndt et.al [10, Theorem 3.1].

3 Main Theorems

In this section, we state and prove our main results.

Theorem 3.1 *Let*

$$\sum_{n=1}^{\infty} c_{15}(n)q^n = 8q^2 f_1^2 f_3^2 f_5^2 f_{15}^2 \left(\frac{747}{w^3} - 45w^3 + 360\right)$$

and

$$\sum_{n=1}^{\infty} c_{(3,5)}(n)q^n = 8q^2 f_1^2 f_3^2 f_5^2 f_{15}^2 \left(\frac{45}{w^3} - 747w^3 + 360\right).$$

Then

$$W_{15}(n) = \frac{\sigma_3(n)}{624} + \frac{3}{208}\sigma_3\left(\frac{n}{3}\right) + \frac{25}{624}\sigma_3\left(\frac{n}{5}\right) + \frac{75}{208}\sigma_3\left(\frac{n}{15}\right) + \left(\frac{1}{24} - \frac{n}{60}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{15}\right) - \frac{c_{15}(n)}{224640} \tag{3.1}$$

and

$$W_{(3,5)}(n) = \frac{\sigma_3(n)}{624} + \frac{3}{208}\sigma_3\left(\frac{n}{3}\right) + \frac{25}{624}\sigma_3\left(\frac{n}{5}\right) + \frac{75}{208}\sigma_3\left(\frac{n}{15}\right) + \left(\frac{1}{24} - \frac{n}{20}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{5}\right) - \frac{c_{(3,5)}(n)}{224640}. \tag{3.2}$$

Proof Set $Z = qf_1 f_3 f_5 f_{15}$. We have

$$-7P_1 + 3P_3 - 5P_5 + 105P_{15} = 7(-P_1 + 3P_3) + 5(-P_5 + 3P_{15}) + 18(-P_3 + 5P_{15}).$$

Replacing q by q^5 in (2.24) and q by q^3 in (2.25) and substituting the resulting identities in the above and using (2.24), we obtain

$$-7P_1 + 3P_3 - 5P_5 + 105P_{15} = 2Z\left[\frac{7}{u}\left(A_1^6 + \frac{27}{A_1^6}\right)^{\frac{2}{3}} + 5u\left(A_5^6 + \frac{27}{A_5^6}\right)^{\frac{2}{3}} + 36v\left(B_3^6 + \frac{125}{B_3^6} + 22\right)^{\frac{1}{2}}\right].$$

Using (2.12), (2.13) and (2.18) in above, we deduce that

$$-7P_1 + 3P_3 - 5P_5 + 105P_{15} = 24Z\left(2\sqrt{t^2+4} + 2t - 1\right).$$

Employing $t = \frac{1}{w^3} - w^3$ above, we obtain

$$-7P_1 + 3P_3 - 5P_5 + 105P_{15} = 24Z\left(\frac{4}{w^3} - 1\right). \tag{3.3}$$

We have

$$-P_1 + 21P_3 - 35P_5 + 15P_{15} = (-P_1 + 3P_3) + 35(-P_5 + 3P_{15}) - 18(-P_3 + 5P_{15}).$$

Replacing q by q^5 in (2.24) and q by q^3 in (2.25) and substituting the resulting identities in the above and using (2.24), we obtain

$$-P_1+21P_3-35P_5+15P_{15}=2Z\left[\frac{1}{u}\left(A_1^6+\frac{27}{A_1^6}\right)^{\frac{2}{3}}+35u\left(A_5^6+\frac{27}{A_5^6}\right)^{\frac{2}{3}}\right.$$
$$\left.-36v\left(B_3^6+\frac{125}{B_3^6}+22\right)^{\frac{1}{2}}\right].$$

Using (2.12), (2.13) and (2.18) in above, we deduce that

$$-P_1+21P_3-35P_5+15P_{15}=24Z\left(2\sqrt{t^2+4}-2t+1\right).$$

Employing $t=\dfrac{1}{w^3}-w^3$ above, we obtain

$$-P_1+21P_3-35P_5+15P_{15}=24Z\left(4w^3+1\right). \tag{3.4}$$

Subtracting (3.4) from seven times (3.3), we have

$$-P_1+15P_{15}=2Z\left(\frac{7}{w^3}-w^3-2\right).$$

Squaring the above identity on both sides, we obtain

$$P_1^2+225P_{15}^2-30P_1P_{15}=4Z^2X^2, \tag{3.5}$$

where

$$X=\left(\frac{7}{w^3}-w^3-2\right).$$

Replacing q by q^{15} in (2.21) and in (2.23) and substituting the resulting identities in the left hand side of (3.5) and using (2.21) and (2.23), we obtain

$$17280\sum_{n=1}^{\infty}\left(\sum_{m<\frac{n}{15}}\sigma(m)\sigma(n-15m)\right)q^n=196+225\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{15}\right)-\frac{96}{5}n\sigma\left(\frac{n}{15}\right)\right)q^n$$
$$+720\sum_{n=1}^{\infty}\sigma\left(\frac{n}{15}\right)q^n+\sum_{n=1}^{\infty}\left(240\sigma_3(n)-288n\sigma(n)\right)q^n$$
$$+720\sum_{n=1}^{\infty}\sigma(n)q^n-4Z^2X^2. \tag{3.6}$$

Consider

$$23Q_1-27Q_3-75Q_5+5175Q_{15}=3\left(Q_1-25Q_5\right)+20\left(Q_1+9Q_3\right)+207\left(25Q_{15}-Q_3\right).$$

Replacing q by q^3 in (2.27) and in (2.28) and substituting the resulting identities along with (2.26), (2.27) and (2.28) in the right hand side of the above, we find that

$$23Q_1-27Q_3-75Q_5+5175Q_{15}=-\frac{72Z^2}{v^2}\left(B_1^6-\frac{125}{B_1^6}\right)+50\left(-P_1+3P_3\right)^2+4968Z^2v^2\left(B_3^6-\frac{125}{B_3^6}\right).$$

Using (2.7), (2.8), (2.19), (2.20) and (2.24) and then using (2.12), we obtain

$$\frac{23}{26}Q_1 - \frac{27}{26}Q_3 - \frac{75}{26}Q_5 + \frac{5175}{26}Q_{15} = 4Z^2X^2 - \frac{1}{13}\sum_{n=1}^{\infty} c_{15}(n)q^n.$$

Using the above in (3.6) to eliminate $4Z^2X^2$ and then using (2.22) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (3.1). Subtracting seven times (3.4) from (3.3), we have

$$-3P_3 + 5P_5 = 2Z\left(\frac{1}{w^3} - 7w^3 - 2\right).$$

Squaring the above identity on both sides, we obtain

$$9P_3^2 + 25P_5^2 - 30P_3P_5 = 4Z^2Y^2, \tag{3.7}$$

where

$$Y = \left(\frac{1}{w^3} - 7w^3 - 2\right).$$

Replacing q by q^3 in (2.21) and in (2.23) and q by q^5 in (2.21) and in (2.23) and substituting the resulting identities in the left hand side of (3.7), we find that

$$\begin{aligned} 17280\sum_{n=1}^{\infty}\left(\sum_{3l+5m=n}\sigma(l)\sigma(m)\right)q^n &= 4 + 9\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{3}\right) - 96n\sigma\left(\frac{n}{3}\right)\right)q^n + 720\sum_{n=1}^{\infty}\sigma\left(\frac{n}{3}\right)q^n \\ &\quad + 25\sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{5}\right) - \frac{288}{5}n\sigma\left(\frac{n}{5}\right)\right)q^n \\ &\quad + 720\sum_{n=1}^{\infty}\sigma\left(\frac{n}{5}\right)q^n - 4Z^2Y^2. \end{aligned} \tag{3.8}$$

Consider

$$-3Q_1 + 207Q_3 + 575Q_5 - 675Q_{15} = 23\left(25Q_5 - Q_1\right) + 20\left(Q_1 + 9Q_3\right) + 27\left(Q_3 - 25Q_{15}\right).$$

Replacing q by q^3 in (2.27) and in (2.28) and substituting the resulting identities along with (2.26), (2.27) and (2.28) in the right hand side of the above, we find that

$$-3Q_1 + 207Q_3 + 575Q_5 - 675Q_{15} = \frac{552Z^2}{v^2}\left(B_1^6 - \frac{125}{B_1^6}\right) + 50\left(-P_1 + 3P_3\right)^2 - 648Z^2v^2\left(B_3^6 - \frac{125}{B_3^6}\right).$$

Using (2.7), (2.8), (2.19), (2.20) and (2.24) and then using (2.12), we obtain

$$-\frac{3}{26}Q_1 + \frac{207}{26}Q_3 + \frac{575}{26}Q_5 - \frac{675}{26}Q_{15} = 4Z^2Y^2 - \frac{1}{13}\sum_{n=1}^{\infty} c_{(3,5)}(n)q^n.$$

Using the above in (3.8) to eliminate $4Z^2Y^2$ and then using (2.22) and then equating the coefficients of q^n on both sides of the resulting identity, we obtain (3.2). □

Note: Using the definition of f_n and the geometric series of $\frac{1}{1-q}$, we have

1. $\frac{Z^2}{w^3} = q\frac{f_3^5 f_5^5}{f_1 f_{15}} = q\prod_{n=1}^{\infty}\left(1-q^{3n}\right)^5\left(1-q^{5n}\right)^5\left(\sum_{k=0}^{\infty}q^{kn}\right)\left(\sum_{k=0}^{\infty}q^{15kn}\right).$

2. $Z^2w^3 = q^3\frac{f_1^5 f_{15}^5}{f_3 f_5} = q^3\prod_{n=1}^{\infty}\left(1-q^n\right)^3\left(1-q^{15n}\right)^5\left(q_2^{\pm 1}, q_2^{\pm 2}, q_1^{\pm 3}, q_2^{\pm 4}, q_1^{\pm 5}, q_1^{\pm 6}, q_2^{\pm 7}; q^{15}\right)_{\infty},$

where

$$(a, b, c, d, e; q)_\infty = (a; q)_\infty (b; q)_\infty (c; q)_\infty (d; q)_\infty (e; q)_\infty$$

and

$$\left(q_m^{\pm t}; q^k\right)_\infty = \prod_{i=0}^{\infty} \left(1 - q^{t+ik}\right)^m \left(1 - q^{k-t+ik}\right)^m.$$

The above confirms that $c_{15}(n)$ and $c_{(3,5)}(n)$ are integers.

4 Applications

Let $\mathbb{Z}$ denote the set of integers and $x_i \in \mathbb{Z}$ for $1 \le i \le 8$. For $b, n \in \mathbb{N}$, let $T_b(n)$ be the number of representations of n by the form

$$\left(x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\right) + b\left(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2\right).$$

Theorem 4.1 *Let*

$$\sum_{n=1}^{\infty} b_5(n)q^n = q^2 f_1^2 f_3^2 f_5^2 f_{15}^2 \left[144\left(\frac{1}{w^3} - w^3\right) + 216\right].$$

Then,

$$T_5(n) = \frac{12}{13}\sigma_3(n) + \frac{108}{13}\sigma_3\left(\frac{n}{3}\right) + \frac{300}{13}\sigma_3\left(\frac{n}{5}\right) + \frac{2700}{13}\sigma_3\left(\frac{n}{15}\right) + \frac{b_5(n)}{13}.$$

Proof For $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, let

$$s_4(k) = \operatorname{card}\left\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 | k = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\right\}.$$

Clearly $s_4(0) = 1$. It is known that [18, Theorem 13, p.266], [20].

$$s_4(k) = 12\sigma(k) - 36\sigma\left(\frac{k}{3}\right), \qquad k \in \mathbb{N}. \tag{4.1}$$

We have

$$\begin{aligned} T_5(n) &= \sum_{\substack{(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\in\mathbb{Z}^8 \\ n=(x_1^2+x_1x_2+x_2^2+x_3^2+x_3x_4+x_4^2)+5(x_5^2+x_5x_6+x_6^2+x_7^2+x_7x_8+x_8^2)}} 1 \\ &= \sum_{\substack{l,m\in\mathbb{Z} \\ l,m\ge 0 \\ l+5m=n}} s_4(l)s_4(m) \\ &= s_4(0)s_4\left(\frac{n}{5}\right) + s_4(n)s_4(0) + \sum_{\substack{l,m\in\mathbb{N} \\ l+5m=n}} s_4(l)s_4(m). \end{aligned}$$

Using (4.1) in the above, we obtain

$$\begin{aligned} T_5(n) = &\left[12\sigma\left(\frac{n}{5}\right) - 36\sigma\left(\frac{n}{15}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right)\right. \\ &\left. + \sum_{\substack{l,m\in\mathbb{N} \\ l+5m=n}} \left(12\sigma(l) - 36\sigma\left(\frac{l}{3}\right)\right)\left(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right)\right)\right] \\ = &\ 12\sigma\left(\frac{n}{5}\right) - 36\sigma\left(\frac{n}{15}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 144\sum_{l+5m=n} \sigma(l)\sigma(m) \\ &-432\sum_{l+5m=n} \sigma\left(\frac{l}{3}\right)\sigma(m) - 432\sum_{l+5m=n} \sigma(l)\sigma\left(\frac{m}{3}\right) + 1296\sum_{l+5m=n} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right) \end{aligned}$$

$$
\begin{aligned}
&= 12\sigma\left(\frac{n}{5}\right) - 36\sigma\left(\frac{n}{15}\right) + 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 144W_5(n) \\
&\quad -432W_{(3,5)}(n) - 432W_{15}(n) + 1296W_5\left(\frac{n}{3}\right).
\end{aligned}
$$

Using the convolution sum $W_5(n)$ from [19, Theorem 1] and employing (3.1) and (3.2) in the above, we find that

$$
\begin{aligned}
T_5(n) = \frac{12}{13}\sigma_3(n) + \frac{108}{13}\sigma_3\left(\frac{n}{3}\right) + \frac{300}{13}\sigma_3\left(\frac{n}{5}\right) + \frac{2700}{13}\sigma_3\left(\frac{n}{15}\right) - \frac{72}{65}\left(9c_5\left(\frac{n}{3}\right) + c_5(n)\right) \\
+\frac{1}{520}\left(c_{15}(n) + c_{(3,5)}(n)\right). \qquad (4.2)
\end{aligned}
$$

It is easy to see that

$$
\sum_{n=1}^{\infty}\left(c_{15}(n) + c_{(3,5)}(n)\right)q^n = 6336Z^2\left(\frac{1}{w^3} - w^3 + \frac{10}{11}\right)
$$

and

$$
\sum_{n=1}^{\infty}\left(9c_5\left(\frac{n}{3}\right) + c_5(n)\right)q^n = Z^2\left(\frac{1}{w^3} - w^3 - 5\right).
$$

Using the above two identities in (4.2), we complete the proof. □

Acknowledgements We are thankful for the valuable suggestions of the anonymous referee for the betterment of this article, in particular for bring our attention to [25] and [26]. The research of the first author is supported by grant F.No.16-6(DEC, 2018)/2019(NET/CSIR) by the funding agency UGC, New Delhi, INDIA and second author is supported by SERB, Govt. of India (EMR/2016/001064).

References

1. A. Alaca, Ş. Alaca, and K. S. Williams. Evaluation of the convolution sums $\sum_{l+12m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+4m=n}\sigma(l)\sigma(m)$. *Adv. Theor. Appl. Math.*, 1(1):27–48, 2006.
2. A. Alaca, Ş. Alaca, and K. S. Williams. Evaluation of the convolution sums $\sum_{l+18m=n}\sigma(l)\sigma(m)$ and $\sum_{2l+9m=n}\sigma(l)\sigma(m)$. *Int. Math. Forum*, 2(1-4):45–68, 2007.
3. A. Alaca, Ş. Alaca, and K. S. Williams. Evaluation of the convolution sums $\sum_{l+24m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+8m=n}\sigma(l)\sigma(m)$. *Math. J. Okayama Univ.*, 49:93–111, 2007.
4. A. Alaca, Ş. Alaca, and K. S. Williams. The convolution sum $\sum_{m<n/16}\sigma(m)\sigma(n-16m)$. *Canad. Math. Bull.*, 51(1):3–14, 2008.
5. Ş. Alaca and Y. Kesicioğlu. Evaluation of the convolution sums $\sum_{l+27m=n}\sigma(l)\sigma(m)$ and $\sum_{l+32m=n}\sigma(l)\sigma(m)$. *Int. J. Number Theory*, 12(1):1–13, 2016.
6. Ş. Alaca and K. S. Williams. Evaluation of the convolution sums $\sum_{l+6m=n}\sigma(l)\sigma(m)$ and $\sum_{2l+3m=n}\sigma(l)\sigma(m)$. *J. Number Theory*, 124(2):491–510, 2007.
7. B. C. Berndt. *Ramanujan's Notebooks: Part II*. Springer New York, 1989.
8. B. C. Berndt. *Ramanujan's Notebooks: Part III*. Springer New York, 1991.
9. B. C. Berndt. *Ramanujan's Notebooks: Part IV*. Springer New York, 1994.
10. B. C. Berndt, H. H. Chan, J. Sohn, and S. H. Son. Eisenstein series in Ramanujan's lost notebook. *Ramanujan J.*, 4(1):81–114, 2000.
11. M. Besge. Extrait d'une lettre de M Besge á M Liouville. *J Math Pure Appl.*, 7:p.256, 1862.
12. H. H. Chan and S. Cooper. Powers of theta functions. *Pacific J. Math.*, 235(1):1–14, 2008.
13. S. Cooper. *Ramanujan's Theta Functions*. Springer International Publishing AG, 2017.
14. S. Cooper and P. C. Toh. Quintic and septic Eisenstein series. *Ramanujan J.*, 19(2):163–181, 2009.
15. S. Cooper and D. Ye. Evaluation of the convolution sums $\sum_{l+20m=n}\sigma(l)\sigma(m)$, $\sum_{4l+5m=n}\sigma(l)\sigma(m)$ and $\sum_{2l+5m=n}\sigma(l)\sigma(m)$. *Int. J. Number Theory*, 10(6):1385–1394, 2014.
16. J. W. L. Glaisher. On the square of the series in which the coefficients are the sums of the divisors of the exponents. *Messenger Math*, 14:156–163, 1862.
17. J. W. L. Glaisher. *Mathematical Papers*, 1883-1885. W. Metcalfe and Son, Cambridge, 1885.
18. J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams. Elementary evaluation of certain convolution sums involving divisor functions. In *Number theory for the millennium, II (Urbana, IL, 2000)*, pages 229–274. A K Peters, Natick, MA, 2002.
19. M. Lemire and K. S. Williams. Evaluation of two convolution sums involving the sum of divisors function. *Bull. Austral. Math. Soc.*, 73(1):107–115, 2006.
20. G. A. Lomadze. Representation of numbers by sums of the quadratic forms $x_1^2 + x_1x_2 + x_2^2$. *Acta Arith.*, 54(1):9–36, 1989.

21. E. Ntienjem. Evaluation of the convolution sums $\sum_{\alpha l+\beta m=n}\sigma(l)\sigma(m)$ where (α, β) is in $\}(1, 14), (2, 7), (1, 26), (2, 13), (1, 28), (4, 7), (1, 30), (2, 15), (3, 10), (5, 6)\{$. Master's Thesis, School of Mathematics and Statistics, Carleton University, 2015.
22. E. Ntienjem. Evaluation of convolution sums involving the sum of divisors function for levels 48 and 64. *Integers*, 17:Paper No. A49, 22, 2017.
23. E. Ntienjem. Evaluation of the convolution sum involving the sum of divisors function for 22, 44 and 52. *Open Math.*, 15(1):446–458, 2017.
24. B. Ramakrishnan and B. Sahu. Evaluation of the convolution sums $\sum_{l+15m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+5m=n}\sigma(l)\sigma(m)$ and an application. *Int. J. Number Theory*, 9(3):799–809, 2013.
25. B. Ramakrishhan and B. Sahu. Evaluation of convolution sums and some remarks on cusp forms of weight 4 and level 12. *Math. J. Okayama Univ.*, 59([2016 on cover]):71–79, 2017.
26. B. Ramakrishnan, B. Sahu, and A. K. Singh. On the number of representations of certain quadratic forms in 8 variables. In *Automorphic forms and related topics*, volume 732 of *Contemp. Math.*, pages 215–224. Amer. Math. Soc., Providence, RI, 2019.
27. S. Ramanujan. *Notebooks (2 volumes)*. Tata Institute of Fundamental Research Bombay, 1957.
28. S. Ramanujan. *The Lost Notebook and Other Unpublished Papers*, Narosa Publishing House, New Delhi, 1988.
29. S. Ramanujan. On certain arithmetical functions [Trans. Cambridge Philos. Soc. **22** (1916), no. 9, 159–184]. In *Collected papers of Srinivasa Ramanujan*, pages 136–162. AMS Chelsea Publ., Providence, RI, 2000.
30. E. Royer. Evaluating convolution sums of the divisor function by quasimodular forms. *Int. J. Number Theory*, 3(2):231–261, 2007.
31. K. S. Williams. The convolution sum $\sum_{m<n/9}\sigma(m)\sigma(n-9m)$. *Int. J. Number Theory*, 1(2):193–205, 2005.
32. K. S. Williams. The convolution sum $\sum_{m<n/8}\sigma(m)\sigma(n-8m)$. *Pacific J. Math.*, 228(2):387–396, 2006.
33. E. X. W. Xia, X. L. Tian, and O. X. M. Yao. Evaluation of the convolution sum $\sum_{i+25j=n}\sigma(i)\sigma(j)$. *Int. J. Number Theory*, 10(6):1421–1430, 2014.
34. D. Ye. Evaluation of the convolution sums $\sum_{l+36m=n}\sigma(l)\sigma(m)$ and $\sum_{4l+9m=n}\sigma(l)\sigma(m)$. *Int. J. Number Theory*, 11(1):171–183, 2015.