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Some Properties of Whole Edge Domination in Graphs

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Abstract: Throughout this paper we initiate a new definition of edge domination in graphs that is called whole edge domination. Some bounds and results for edge whole domination number are established. Moreover, calculate this number of some certain graphs. Also, the effect of deletion, addition, and contraction of an edge on the domination of this definition has been determined.

Keywords: edge dominating set, whole edge dominating set, whole edge domination number.

1 Introduction

Graph theory has now become a new language that deals with all scientific and even literary sciences. It can give another face to prove most scientific problems through its simple tools. One of the most important topics that graph theory deals with is the topic of domination because it has very wide applications in most fields. dominance is. Emerged many definitions that solve the problems of life by placing restrictions on the dominating set or outside the dominating set or in both. C.Berge is the first appearance of this concept was in [1], and the first to deal with this concept is Ore in his book[2]. Currently, many new definitions are appeared, as in [3,4,5,6,7,8,9]. In mathematics, in particular, this concept deals with various fields such as fuzzy graph[10,11,12,13,14], topological graph [15], topological indices[16,17,18,19,20,21,22] and others. There are two methods for calculating the domination in graphs, the first is by the set of vertices and the second is by the set of edges. In this work, the domination will be calculated by means of the set of edges [23,24,25,26,27,28,29]. In this work, a new definition of edge domination is been introduced, which is the whole edge domination in graphs. Many bounds and properties of this definition have been determined. Moreover, for certain graphs, this number has been introduced. Finally, the effect of deletion, addition, and contraction of an edge on the domination of this definition has been calculated. For a comprehensive bibliography of papers on the concept of domination, readers are referred to F. Harary [30].

Definition 1.[5] *Two edges e_1 and e_2 of G are called adjacent if they are distinct and have a common end-vertex. The open neighborhood $\mathcal{N}_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its closed neighborhood is $\mathcal{N}_G[e] = \mathcal{N}_G(e) \cup e$.*

Definition 2.[2] *A set $X \subseteq E$ is said to be an edge dominating set if every edge in $E - X$ is adjacent to some edge in X. The edge domination number of G is the cardinality of a smallest edge dominating set of G and is denoted by γ'.*

Definition 3.[5] *A graph H is a subgraph of G if , $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that a subgraph H is a spanning subgraph of G if H contains all vertices of G.*

Definition 4.[5] *The complement $\bar{G}$ of a simple graph G with vertex set $V(G)$ is the graph in which two vertices are adjacent if and only if they are not adjacent inG*

Definition 5.[5] *$G - e$ is the graph obtained from G by deleting the edge e where e is an edge in G. Similarly, if v is a $G - v$ is the graph obtained from G by deleting the vertex v together with the edges incident onv where, v vertex of G.*

We can find all other concepts in [5]. For more details on several advanced topics about domination in graphs see[1,

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2,3,4,6,7,9]. Here, new definition of domination number in graph is introduced. Properties and some bounds for whole edge domination are determined. Also, the effects on $\gamma_{whe}(G)$ when modified G are discussed.

2 Whole edge domination number

Now, we initiate a new definition of edge dominating set in a graph and we call it a whole edge dominating set. Also, some dimension of whole edge dominating set are mentioned.

Definition 6. *In a graph $G=(V,E)$, a proper subset $X\subseteq E$ is called whole edge dominating set $(WEDS)$, if every edge in X is adjacent to all edges in $E-X$.*

Definition 7. *In a graph $G=(V,E)$, If X is a $WDES$, then X is called minimal $WDES$, if it has no proper $WDES$.*

Definition 8. *A minimal WEDS has smallest cardinality is called whole edge domination number denoted by $\gamma_{whe}(G)$.*
In the next two definitions the effects on $\gamma_{whe}(G)$ are discussed.

Definition 9. *Let $G=(V,E)$ be a graph and $e\in E$, when we delete an edge e from G then the edges of G are partition into three sets*

$$E_{-}^{0}=\{e\in E,\ \gamma_{whe}(G-e)=\gamma(G)\},$$
$$E_{-}^{-}=\{e\in E,\ \gamma_{whe}(G-e)<\gamma_{whe}(G)\},$$
$$E_{-}^{+}=\{e\in E,\ \gamma_{whe}(G-e)>\gamma(G)\}.$$

Definition 10. *Let $G=(V,E)$ be a graph and $e\in\bar{\bar{G}}$, when an edge e addition to G,edges set can be partitioned into tree sets*

$$E_{+}^{0}=\{e\in E,\ \gamma_{whe}(G+e)=\gamma_{whe}(G)\},$$
$$E_{+}^{-}=\{e\in E,\ \gamma_{whe}(G+e)<\gamma_{whe}(G)\},$$
$$E_{+}^{-}=\{e\in E,\ \gamma_{whe}(G+e)>\gamma_{whe}(G)\}.$$

Remark. If a graph $G=(V,E)$ is disconnected , then G has no whole edge dominating set.

Observation 21

1. *For the path graph P_n with $n\geq 3$,*
$$\gamma_{whe}(P_n)=\begin{cases}1,\ \text{if } 3\leq n\leq 4;\\ P_n \text{ has no } WDES,\ Otherwise.\end{cases}$$
2. *For the cycle graph C_n with $n\geq 3$,*
$$\gamma_{whe}(C_n)=\begin{cases}1, \text{if } n=3;\\ 2,\ \text{if } n=4;\\ C_n, \text{has no } WDES\ Otherwise.\end{cases}$$
3. *For the complete graph K_n with $n\geq 3$,*
$$\gamma_{whe}(K_n)=\begin{cases}1, \text{if } n=3;\\ 2,\ \text{if } n=4;\\ K_n, \text{has no } WDES,\ Otherwise.\end{cases}$$
4. *For a Wheel graph W_n with $n\geq 3$,*
$$\gamma_{whe}(W_n)=\begin{cases}2,\ \text{if } n=3;\\ W_n \text{ has no } WDES,\ Otherwise.\end{cases}$$
5. *For star graph S_n with $n\geq 3$,*
$$\gamma_{whe}(S_n)=1$$
.

Theorem 1. *Tree has a whole edge domination if and only if $2\leq Daim(T)\leq 3$.*

Proof. Let T be a tree and let X be a $WEDS$ with minimum cardinality. Suppose that X contains two edges say $\{e_1,e_2\}$, then there are two cases as follows.
Case1: If the number of the remained edges in $G-X$ is one say e_3 then since the graph is a tree so it must be a path of order four with an edge e_3 which the encident vertices on it of $degree=2$. Thus, $\{e_3\}$ is a $WEDS$ and this is a contradiction with the set X is the minimum set.
Case2: If the number of the remained edges in $G-X$ more than one, then there is a cycle contains the edges in X and the other edges. Again, this is a contradiction with assumption.
Therefore, X has one edge, so $2\leq Daim(T)\leq 3$.
Conversely, if $2\leq Daim(T)\leq 3$, then the middal edge in this tree is dominating to all edges. Thus, the required is satisfied.

Corollary 1. *If a graph $G=(V,E)$ be a double star $S_{m,n}$. Then*

$$\gamma_{whe}(S_{m,n})=1.$$

Theorem 2. *If a graph G has domination number γ_{whe}, then*
$1\leq\gamma_{whe}(G)\leq 2$, furthermore

1. *if $\gamma_{whe}(G)=1$, then G contains a spanning subgraph either star or double star.*
2. *if $\gamma_{whe}(G)=2$, then $G=C_4$ or K_4.*

Proof. Let X be a minmal dominating set with minimum cardinality. Suppose that $|X|=3$, then the following cases are arise.
i) If all edges in X are adjacent, then $G[X]$ is a complete graph of order three. In this case we cannot take all edges in X, since if all vertices of these edges in X are adjacent to other edges not in X, then there is no $WEDS$, since for each edge in X there are at least one edge not adjacent to it. Otherwise $\gamma_{whe}=1$ and this is a contradiction with minimality of the set X.

ii) If $G[X]$ is not a complete graph, and connected, then $G[X]$ is a path of order4. Now if all edges are adjacent to the middale edge then $\gamma_{whe}(G)=1$. Otherwise $Daim(G)\geq 4$, and this is a contradiction.
Similrly if $|X|>3$,.
For all cases above, we obtain contradiction.Therefore,

$1 \leq |X| \leq 2$. Which means that $1 \leq \gamma_{whe} \leq 2$.
Now to proof 1 and 2
1) If $\gamma_{whe}(G) = 1$, then there is an edge that is adjacent to all edges in G. Thus, if this edge is adjacent to all edges with one common vertex, then there is a spanning subgraph isomorphic to star. If this edge is adjacent to all edges by its two incedent vertices, then there is a spanning subgraph isomorphic to double star.
2) If $\gamma_{whe}(G) = 2$, then all edges which are not in X are adjacent to two edges in X say e_1 *and* e_2. Now, if there is one edge which is not in X it is adjacent to these two edges, then this edge is a whole edge dominating set and this is a contradiction with $\gamma_{whe} = 2$. If there are two edges that are adjacent to e_1 and e_2. such that these edges are adjacent to either e_1 *or* e_2 by the same vertex, so one of the edges is not in X and it is a WEDS and again this is a contradiction. Therefore, the graph e_1, e_2 and the other two edges must be a cycle or complete of order four. Thus, we get the result.

Theorem 3. *If a graph $G = (V,E)$ of size m and order n has a whole domination number γ_{whe}, then*

$$2 \leq m \leq \frac{n(n-1)}{2} if\ n = 3,4. \quad (1)$$

$$2 \leq m \leq 2n-3. \quad (2)$$

Proof. Let X be a $WDES$ with minimum cardinality. We get the lower bound when $G = P_3$ and the upper bound by the following two cases

1. If G is a complete graph then, $2 \leq m \leq \frac{n(n-1)}{2}$ if $n = 3,4$
2. If G is a cycle of order four, then $m = 4 = n$.

i) If G is a cycle of order four, then $m = 4 = n$.
ii) If $\gamma_{whe}(G) = 1$, and G contains a spanning subgraph that it is isomorphic to star, then there is an edge e=uv such that it is adjacent to all edges in G by one of its two incident vertices say v and the number of edges that is incident with v is $n-1$. Also, the vertex u can be adjacent to all vertices of G and the number of edges in this case is $n-2$. Thus, the maximum number of edges is $n-1+n-2 = 2n-3$.
iii) If $\gamma_{whe}(G) = 1$, and G contains a spanning subgraph isomorphic to double star , so there is an edge e=uv that is adjacent to all edges in$G$ such that the two vertices of e, v and u are adjacent to $n-2$ vertices, so the maximum number of edges in this case is $2(n-2)+1 = 2n-3$. For all cases in 2 we get $2 \leq m \leq\ 2n-3$. Therefore, we get the result.

Theorem 4. *If a graph $G = (V,E)$ has a whole edge domination number γ_{whe}, then E_+^0, E_+^- and E_+^+ are not empty sets*

Proof. Two cases are appear as follows.
Case 1 If a graph G is a star, then $\gamma_{whe}(G+e) = \ \gamma_{whe}(G)$. So, $E_+^0 \neq \emptyset$
Case 2 If there is no edge in X is adjacent to the addition an edge e, then two cases are appear as follows.
i) If we take $G = P_4$ and add the edge that is incident to the two pendants vertices of P_4, so $\mathbb{G}+e = C_4$, therefore $\gamma_{whe}(G+e) > \ \gamma_{whe}(G)$ *.So,* $\mathbb{E}_+^+ \neq \emptyset$
ii) If we take$C(G) = C_4$ and add an edge incident to the vertices which are not adjacent in C_4D, so, $G+e$ contains an edge adjacent to all edges. Thus, $\gamma_{whe}(G+e) < \ \gamma_{whe}(G)$. Hence, $E_+^- \neq \emptyset$.

Theorem 5. *If a graph $G = (V,E)$ has γ_{whe}, then E_-^0, E_-^- and E_-^+ are not empty sets.*

Proof. As same manner in Theorem 2.5 by deleting the adding edge

Theorem 6. *If G has $\gamma_{\ whe}$, then $\gamma_{whe}(G-v) \leq \ \gamma_{whe}(G)$ where $v \in V$ or $G-v$ has no WEDS.*

Proof. There are two cases as follows.
Case1. If $G-v$ is disconnected, then $G-v$ has no whole dominating set.
Case 2. If $G-v$ is connected, then by applying Theorem 2.4 there are two cases as follows.
i) If $\gamma_{whe}(G) = 1$, then G includes a spanning subgraph either it is a star or double star. Now, if G includes a spanning subgraph isometric to star, then there are two cases as follows.
a) If a graph G has maximum number of edges, which means there is an edge say $e = vu$, such that u and v are adjacent to all other vertices. Thus, if we delete any vertex from this graph the whole edge dominating is not influenced by this deletion, that means $\gamma_{whe}(G-v) = \ \gamma_{whe}(G)$ (as an example, see figure 2.1).

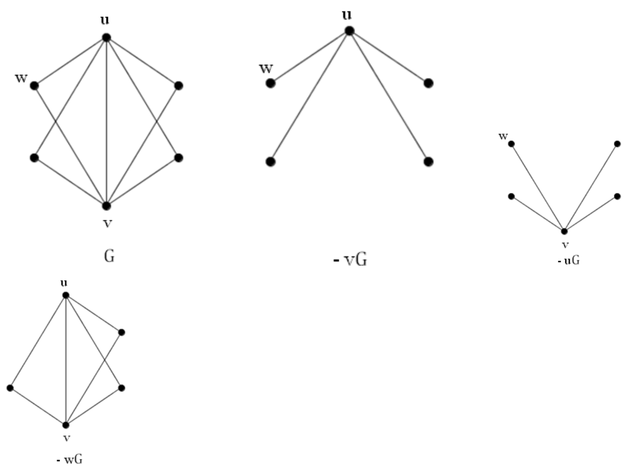

Figure 2.1 examples show the proof theorem 6 (a)

b). If a graph G has no maximum number of edges, which means the vertex u that is incident with the edge e is not adjacent to some other vertices in G, so if we delete the vertex v, then we get the an isolated vertex, so $G-v$ has no whole edge dominating set. Otherwise, deleting any vertex from graph G do not influence the whole edge domination (as an example, see Figure 2.2) .

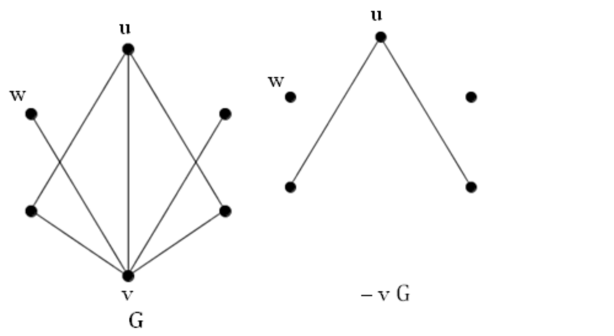

Figure 2.2 If a graph G has no maximum number of edges

Now, if G contains a spanning subgraph isometric to double star, then there are two cases as follows.
c). If $e = vu$ is the edge that dominate all edges in G, and if we delete u or v, then the graph $G-v$ or G-u has an isolated vertices. Thus $G-v$ or $G-u$ has on whole edge dominating set (as an example, see Figure 2.3)
D). If the deleted vertex is not adjacent to the edge e which is dominating the graph edges, then this deletion do not influence to whole edge domination edge.

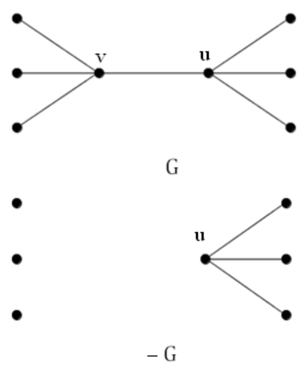

Figure 2.3 examples show the proof theorem 6 (c)

ii) If $\gamma_{whe}(G) = 2$, then $G = C_4$ or K_4. Thus, $G-v$ is a path of order three, so $\gamma_{whe}(\mathbb{G}-v) = 1$.

Therefore, $\gamma_{whe}(G-v) < \gamma_{whe}(G)$.

For all cases above, one can see that $\gamma_{whe}(G-v) \leq \gamma_{whe}(G)$ or $G-v$ has no whole edge dominating set.

Theorem 7.*If a graph $G = (V,G)$ has γ_{whe}. Then*

$$\gamma_{whe}(G/e) \leq \gamma_{whe}(G).$$

*Proof.***Case1**. If e belongs to the γ_{whe-}set, then by Theorem 2.4 we have two cases as follows.
i) If $\gamma_{whe}(G) = 1$, then G includes a spanning subgraph either star or double star. Now if G includes a spanning subgraph isomorphic to double star, then the edge e join the two stars. Thus, when we contract this edge the graph becomes isomorphic to star.Therefore, $\gamma_{whe}(G\ e) = \gamma_{whe}(G)$. In the same manner if G contains a spanning subgraph isomorphic to star. This graph becomes a star too when we contract the edge e.
ii) If $\gamma_{whe}(G) = 2$, then $G \cong C_4$ or K_4. Then when we contract the edge e graph G turn to C_3, which means the whole edge domination is equal to one. Therefore, in this case $\gamma_{whe}(G\ e) < \gamma_{whe}(G)$.
Case 2. If e does not belong to any γ_{whe-}set, then the contract edge not influence whole domination number.
bfCase 2. If e does not belong to any γ_{whe-}set, then contracting an edge e do not influence to whole domination number of G. Therefore, we get the result.

*Remark.*If $G-v$ has whole edge domination number, then G is not necessary has whole edge domination number (as an example see Fig. 2.4)

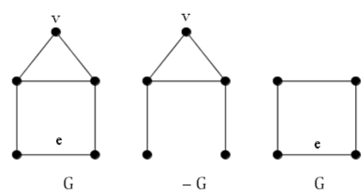

Figure 2.4 If whole edge if $G-v$ has whole edge domination number

2) If $G-e$ has a whole edge domination number, then G is not necessary has whole edge domination number (as an example see Fig. 2.4).
3) If $G+e$ has whole edge domination number, then G not necessary has whole edge domination number (as an example see Fig. 2.5).

4) If $G\ e$has a whole edge domination number, then G is not necessary has a whole edge domination number (as an example see Fig. 2.6).

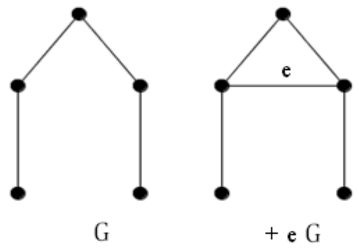

Figure 2.5 whole edge if $G+e$ has whole edge domination number

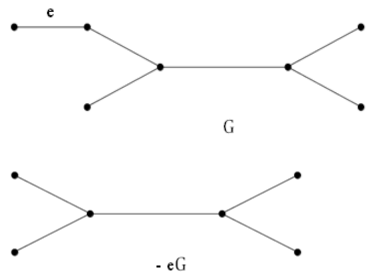

Figure 2.6 whole edge If $G\ e$ has a whole edge domination number

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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