On some continued fraction expansions for the ratios of the function $\rho(a, b)$

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Abstract. In his lost notebook, Ramanujan has defined the function $\rho(a, b)$ by

$$\rho(a, b) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n},$$

where $|q| < 1$, and $(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$, $n = 1, 2, 3, \ldots$, and has given a beautiful reciprocity theorem involving $\rho(a, b)$. In this paper we obtain some continued fraction expansions for the ratios of $\rho(a, b)$ with some of its contiguous functions. We also obtain some interesting special cases of our continued fraction expansions which are analogous to the continued fraction identities stated by Ramanujan.

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1 Introduction

Ramanujan, a pioneer in the theory of continued fractions has recorded scores of continued fraction identities in chapter 12 of his second notebook [23] and in his lost notebook [24]. This part of Ramanujan’s work has been treated and developed by several authors including Andrews [4], Hirschhorn [19], Carlitz [12], Gordon [18], Al-Salam and Ismail [3], Ramanathan [21], [22], Denis
The main purpose of this paper is to establish continued fraction expansions for the ratios \( \rho(aq, b)/\rho(a, b) \) and \( \rho(a, bq)/\rho(a, b) \), where

\[
\rho(a, b) = \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n}, \tag{1.1}
\]

which was given by Ramanujan in his lost notebook [24]. In fact Ramanujan has given a beautiful reciprocity theorem for the function \( \rho(a, b) \) in his lost notebook.

In section 2, we prove some key functional relations satisfied by \( \rho(a, b) \), which will be used in the development of continued fractions. In section 3, we prove our main results and in section 4 we obtain some special cases of our continued fractions which are analogous to the continued fractions of Ramanujan.

## 2 Some functional relations satisfied by \( \rho(a, b) \)

In this section, we prove some functional relations satisfied by \( \rho(a, b) \).

**Lemma 1.** \( \rho(a, b) \) satisfies the following functional relations.

\[
(1 + aq) \frac{\rho(a, b)}{(1 + \frac{1}{b})} - aq \frac{\rho(aq, b)}{(1 + \frac{1}{b})} = \frac{\rho(aq, bq)}{(1 + \frac{1}{bq})}, \tag{2.1}
\]

\[
(1 + aq) \frac{\rho(a, bq)}{(1 + \frac{1}{bq})} - (1 + aq) \frac{\rho(a, b)}{(1 + \frac{1}{b})} = \frac{aq \rho(aq, b)}{b} - \frac{a \rho(aq, bq)}{b (1 + \frac{1}{b})}, \tag{2.2}
\]

\[
\frac{\rho(a, bq)}{(1 + \frac{1}{bq})} = \left(1 - \frac{a}{b}\right) \frac{\rho(a, b)}{(1 + \frac{1}{b})} + \frac{aq}{b} \frac{(1 + a) \rho(aq, b)}{(1 + \frac{1}{b})}, \tag{2.3}
\]

\[
\rho(a, b) = \left(1 - \frac{aq}{1 + aq}\right) \rho(aq, b) + \left(\frac{aq^2/b}{1 + aq^2}\right) \rho(aq^2, b), \tag{2.4}
\]

\[
(1 + aq) \frac{\rho(a, bq)}{(1 + \frac{1}{bq})} - aq \frac{\rho(aq, bq)}{(1 + \frac{1}{bq})} = \frac{\rho(aq, bq^2)}{(1 + \frac{1}{bq^2})}, \tag{2.5}
\]

\[
\rho(a, b) = \left[\frac{a + bq(a - 1)}{a(1 + bq)}\right] \rho(a, bq) + \left[\frac{bq}{a(1 + bq^2)}\right] \rho(a, bq^2) \tag{2.6}
\]
Proof. Using (1.1), the left side of (2.1) can be written as

\[
(1 + aq) + (1 + aq) \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n}
\]

\[
- aq - aq \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (aq) b^n}{(-aq^2)_n}
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq^2)_{n-1}} \left\{ 1 - \frac{aq^{n+1}}{1 + aq^{n+1}} \right\} = \rho(aq, bq) \frac{1}{1 + \frac{aq}{bq}},
\]

which is the right side of (2.1).

Using (1.1), the left side of (2.2) can be written as

\[
(1 + aq) \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n (bq)^{-n}}{(-aq)_{n}} - (1 + aq) \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_{n}}
\]

\[
= -\frac{a}{b} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n-1)/2} a^{n-1} b^{-n+1}}{(-aq^2)_{n-1}}
+ \frac{aq}{b} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n-1)/2} (aq)^{n-1} b^{-n+1}}{(-aq^2)_{n-1}}
\]

\[
= \frac{aq}{b} \rho(aq, bq) \frac{a}{b} \rho(aq, bq) \frac{1}{1 + \frac{aq}{bq}}.
\]

This proves (2.2).

Substituting for \(\rho(aq, bq)/(1 + 1/bq)\) in (2.2) from (2.1), we obtain (2.3) on some simplifications.

Changing \(a\) to \(aq\) in (2.3), then adding resulting equation to (2.1), we obtain (2.4).

Using (1.1), the left side of (2.5) can be written as

\[
(1 + aq) + (1 + aq) \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n (bq)^{-n}}{(-aq)_{n}}
\]

\[
- aq - aq \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (aq) b^n}{(-aq^2)_n}
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n (bq)^{-n}}{(-aq^2)_{n-1}} \left\{ 1 - \frac{aq^{n+1}}{1 + aq^{n+1}} \right\} = \rho(aq, bq^2) \frac{1}{1 + \frac{aq}{bq^2}},
\]

which is the right side of (2.5).
Adding (2.1), (2.2) and the negative of (2.5), we obtain (2.6) on some simplifications.

3 Main results

In this section, we deduce the continued fraction expansions for the ratios \( \rho(aq, b)/\rho(a, b) \) and \( \rho(a, bq)/\rho(a, b) \).

**Theorem 1.** We have

\[
\frac{\rho(aq, b)}{\rho(a, b)} = \frac{(1 + aq)}{N_1 + \frac{M_1}{N_2 + \frac{M_2}{N_3 + \cdots \frac{M_n}{N_{n+1} \cdots}}}} \tag{3.1}
\]

where

\[
M_n = \frac{aq^{n+1}}{b} (1 + aq^n),
\]

and

\[
N_n = \left(1 - \frac{aq^n}{b} + aq^n\right), \quad n = 0, 1, 2, \ldots
\]

**Proof.** Changing \( a \) to \( aq^n \) in (2.4), we obtain

\[
\rho(aq^n, b) = \left(1 - \frac{aq^n+1}{b} + aq^{n+1}\right) \rho(aq^{n+1}, b) + \left(\frac{aq^{n+2}/b}{1 + aq^{n+2}}\right) \rho(aq^{n+2}, b).
\]

This can be written as

\[
T_n = \frac{\rho(aq^n, b)}{\rho(aq^{n+1}, b)} = \left(1 - \frac{aq^n+1}{b} + aq^{n+1}\right) + \left(\frac{aq^{n+2}/b}{1 + aq^{n+2}}\right) \frac{T_{n+1}}{T_{n+1}}. \tag{3.2}
\]

Iterating (3.2) with \( n = 0, 1, 2, \ldots \), and then taking reciprocals, we obtain (3.1) after some simplifications.

**Theorem 2.** We have

\[
\frac{\rho(a, bq)}{\rho(a, b)} = \frac{(1 - \frac{a}{b})(1 + bq)}{q(1 + b) + \frac{(1 + bq)M_0}{q(1 + b)N_1 + \frac{M_1}{N_2 + \cdots \frac{M_n}{N_{n+1} \cdots}}}} \tag{3.3}
\]

where \( M_n \) and \( N_n \) are as in theorem (3.1).

**Proof.** Equation (2.3) can be written as

\[
\frac{\rho(a, bq)}{\rho(a, b)} = \frac{(1 - \frac{a}{b})(1 + bq)}{q(1 + b) + \frac{aq(1 + a)(1 + bq)}{q(1 + b)(1 + aq) \frac{\rho(aq, b)}{\rho(a, b)}}}. \tag{3.4}
\]

Iterating (3.2) with \( n = 0, 1, 2, \ldots \), and substituting the resulting identity in (3.4), we obtain (3.3) after some simplifications.
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**Theorem 3.** We have

$$\frac{\rho(a, bq)}{\rho(a, b)} = \frac{a(1 + bq)}{A_0} \frac{B_0}{A_1} \frac{B_1}{A_2} \cdots \frac{B_n}{A_{n+1}} \cdots, \quad (3.5)$$

where

$$A_n = [a + bq^{n+1}(a - 1)],$$

and

$$B_n = [abq^{n+2}(1 + bq^{n+1})], \quad n = 0, 1, 2, \ldots.$$  \(\text{Proof.} \) Changing $b$ to $bq^n$ in (2.6), we obtain on some simplifications

$$\rho(a, bq^n) = \left[\frac{a + bq^{n+1}(a - 1)}{a(1 + bq^{n+1})}\right]\rho(a, bq^{n+1}) + \left[\frac{bq^{n+2}}{a(1 + bq^{n+2})}\right]\rho(a, bq^{n+2}).$$

This can be written as

$$F_n \equiv \frac{\rho(a, bq^n)}{\rho(a, bq^{n+1})} = \left[\frac{a + bq^{n+1}(a - 1)}{a(1 + bq^{n+1})}\right] + \left[\frac{bq^{n+2}}{a(1 + bq^{n+2})}\right]. \quad (3.6)$$

Iterating (3.6) with $n = 0, 1, 2, \ldots$, and then taking reciprocals, we obtain (3.5) after some simplifications.  \(\square\)

4 Some special cases

In this section, we derive the following special cases of (3.1), (3.3) and (3.5).

$$\sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+3)/2}}{(q)_{n}} = \frac{1 + q}{1 + a(1 + q^2)} \frac{q^2(1 + q)}{1 + \cdots}, \quad (4.1)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}}{(q)_{n}} = \frac{1 - q}{1 - a(1 + q^2)} \frac{q^2(1 - q)}{1 - \cdots}, \quad (4.2)$$

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)/2}}{(q)_{n}} = \frac{1 - q}{2 - q - (1 + q^2)} \frac{q(1 - q)}{1 - \cdots}, \quad \sum_{n=0}^{\infty} \frac{q^{(n-1)/2}}{(q)_{n}} = \frac{1 - q}{2 - q - (1 + q^2)} - \cdots, \quad (4.3)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+5)/2}}{(-q^2)_{n}} = \frac{1 + q^2}{1 + a(1 + q^2)} \frac{q^3(1 + q^2)}{1 + \cdots}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+5)/2}}{(-q^2)_{n}} = \frac{1 + q^2}{1 + a(1 + q^2)} \frac{q^3(1 + q^2)}{1 + \cdots}, \quad (4.4)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+5)/2}}{(-q^2)_{n}} = \frac{1 + q^2}{1 + a(1 + q^2)} \frac{q^3(1 + q^2)}{1 + \cdots}, \quad (4.4)$$
\[
\sum_{n=0}^{\infty} \frac{q^n(n-1)/2}{(q^2)_n} = 2 - \frac{q(1 - q)}{(1 + q - q^2) - (1 + q^2 - q^3)} - \ldots, \tag{4.5}
\]
\[
\sum_{n=0}^{\infty} \frac{q^n(n+1)/2}{(q^2)_n} = (1 + q) - \frac{q^2(1 - q)}{1 - q^3(1 - q^2)} - \ldots, \tag{4.6}
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^n(n-3)/2}{(-q^2)_n} = \frac{q - 1}{q} + \frac{(1 + q)}{q^2 + \ldots}, \tag{4.7}
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^n(n+1)/2}{(-q^2)_n} = (1 - q) + \frac{q^2(1 + q^2)}{(1 - q^4 + q^3)} + \ldots, \tag{4.8}
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^n(n-1)/2}{(-q^2)_n} = 2q \cdot \frac{q^2(1 + q)}{1 + \frac{q^3(1 + q^2)}{1 + \ldots}}, \tag{4.9}
\]
\[
\sum_{n=0}^{\infty} \frac{q^n(n-1)/2}{(q^2)_n} = \frac{2q}{1 + 2q} - \frac{q^2(1 + q)}{1 + 2q^2} - \ldots, \tag{4.10}
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^n(n+1)/2}{(-q^2)_n} = 2 \cdot \frac{(1 + q)}{1 + (1 - q + q^2)} + \frac{q^2(1 + q^2)}{(1 - q^2 + q^3)} + \ldots, \tag{4.11}
\]
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^n(n-3)/2}{(-q^2)_n} = q(1 + q) \cdot \frac{q^2(1 + q^2)}{1 + q^4(1 + q^3)} \ldots. \tag{4.12}
\]

**Proof.** Setting \( a = 1 = b \) in (3.1) and using the definition (1.1) of \( \rho(a, b) \) we obtain (4.1) after some simplifications. Similarly we obtain (4.2), (4.3) and (4.4) from (3.1) for \( a = -1, b = 1; a = -1, b = q \) and \( a = q, b = 1 \) respectively.

Setting \( a = -q, b = q \) in (3.3) and using the definition (1.1) of \( \rho(a, b) \) we obtain (4.5) after some simplifications. Similarly we obtain (4.6), (4.7) and (4.8) from (3.3) for \( a = -q, b = 1; a = q, b = q^2 \) and \( a = q^2, b = q \) respectively.

Setting \( a = 1 = b \) in (3.5) and using the definition (1.1) of \( \rho(a, b) \) we obtain (4.9) after some simplifications. Similarly we obtain (4.10), (4.11) and (4.12) from (3.5) for \( a = -1, b = 1; a = q, b = 1 \) and \( a = 1, b = q \) respectively. \(\square\)

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