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A generalization of chromatic index

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Abstract

Let $G=(V,E)$ be a graph and $k \geqslant 2$ an integer. The *general chromatic index* $\chi_k'(G)$ of G is the minimum order of a partition P of E such that for any set F in P every component in the subgraph $\langle F \rangle$ induced by F has size at most $k-1$. This paper initiates a study of $\chi_k'(G)$ and generalizes some known results on chromatic index.

The purpose of this paper is to obtain a generalization of chromatic index. Compared to many generalizations of chromatic number, there exist very few generalizations of chromatic index in the literature. For example, see [2] and [3].

Let $G=(V,E)$ be a graph and $k \geqslant 2$ an integer. A set $F \subset E$ is and I_k*-set* (or *k-independent set*) if every component in the subgraph $\langle F \rangle$ induced by F has size at most $k-1$. Equivalently, a set $F \subset E$ is k-independent if the sum of the degrees of the vertices in every component of the subgraph $\langle F \rangle$ is r, where $2 \leqslant r \leqslant 2(k-1)$.

A partition $\{E_1, E_2, \ldots, E_r\}$ of E is an I_k*-partition* if each E_i is an I_k-set. An I_k*-edge coloring* of G is a coloring of the edges of G so that the set of all edges receiving the same color is an I_k-set. An I_k-edge coloring which uses r colors is called a (k,r)*-edge coloring*.

The *k-chromatic index* $\chi_k' = \chi_k'(G)$ of G is the minimum number of colors needed in an I_k-edge coloring of G. If $\chi_k'(G)=n$, then G is said to be (k, n)*-edge chromatic*. The *k-edge independence number* $\beta_{1k}=\beta_{1k}(G)$ of G is the maximum cardinality of an I_k-set. Clearly, if M is any independent set of edges, then M is an I_k-set for all $k \geqslant 2$.

We observe that $\chi_2'(G)=\chi'(G)$, the chromatic index. Also $\beta_{12}=\beta_1$, the edge independence number of G. If G has size q, then $\chi_k'(G)=1$ for all $k>q$. If $L(G)$ is the line graph of G, then

$$\chi'(G)=\chi(L(G)) \tag{1}$$

where $\chi(L(G))$ is the chromatic number of $L(G)$.

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$\chi'_k(K_p)$

k \ p	3	4	5	6	7	8	9
3	2	3	4	4	6	6	7
4	1	2	3	4	4	4	4
5	1	2	3	3	4	4	4
6	1	2	2	3	3	3	4
7	1	1	2	3	3	3	4
8	1	1	2	3	3	3	4
9	1	1	2	2	3	3	3

Fig. 1.

$\chi'_k(K_{n,n})$

k \ n	3	4	5	6	7	8
3	3	4	5	6	7	8
4	3	3	4	5	6	6
5	2	2	3	4	4	4
6	2	2	3	4	4	4
7	2	2	3	3	4	4
8	2	2	3	3	4	4
9	2	2	3	3	4	4

Fig. 2.

The vertex analogue of $\chi'_k(G)$ has been defined by Sampathkumar [5] as follows: Let $k \geqslant 2$ be an integer. The k-chromatic number $\chi_k(G)$ of G is the minimum order of a partition $\{V_1, V_2, \ldots, V_n\}$ of V such that every component in the subgraph $\langle V_i \rangle$ induced by V_i has order at most $k-1$. Clearly, for any graph G with size $q \geqslant 1$

$$\chi'_k(G) = \chi_k(L(G)) \tag{2}$$

The problem of determining the k-chromatic index for the complete graph K_p and the complete bipartite graph $K_{m,n}$ is open. However Figs. 1 and 2 will give the k-chromatic index of these graphs in some cases.

Also $\chi'_{10}(K_7) = 3$

$\chi'_k(K_7) = 2$, for $11 \leqslant k \leqslant 21$

$\chi'_k(K_8) = 3$, for $10 \leqslant k \leqslant 14$

$= 2$, for $15 \leqslant k \leqslant 28$

$\chi'_k(K_9) = 3$, for $10 \leqslant k \leqslant 18$

$= 2$, for $19 \leqslant k \leqslant 36$

Also $\chi'_k(K_{7,7}) = 3$, for $k = 10, 11$

$\chi'_k(K_{8,8}) = 4$, for $k = 10, 11$

$\chi'_k(K_{n,n}) = 3$, $12 \leqslant k \leqslant 16$, $n = 7, 8$.

$\chi'_k(K_{n,n}) = 2$, $10 \leqslant k \leqslant 16$, $4 \leqslant n \leqslant 6$.

$= 2$, $17 \leqslant k \leqslant 25$, $5 \leqslant n \leqslant 8$.

$= 2$, $26 \leqslant k \leqslant 36$, $6 \leqslant n \leqslant 8$.

$= 2$, $37 \leqslant k \leqslant 49$, $n = 7, 8$.

$\chi'_k(K_{8,8}) = 2$, $50 \leqslant k \leqslant 64$.

Let G be a graph of order p, and $2 \leqslant k \leqslant r$. If G is a cycle, then $\chi'_k(G) = 2$. We also observe that for all $2 \leqslant k \leqslant r$, an I_k-set is an I_r-set, and

$$\beta_1 = \beta_{12} \leqslant \beta_{1k} \leqslant \beta_{1r}, \tag{3}$$

$$\chi'_r \leqslant \chi'_k \leqslant \chi'_2 = \chi'. \tag{4}$$

Proposition 1. *For any graph* $G=(V,E)$, (i) $\beta_{1k}\leqslant(k-1)\beta_1$, *and* (ii) $\chi'\leqslant(k-1)\chi'_k$.

Proof. (i) Let $F\subset E$ be an I_k-set with $|F|=\beta_{1k}$. Clearly, the subgraph $\langle F\rangle$ can contain at most β_1 components, and each component containing at most $k-1$ edges. Thus $|F|=\beta_{1k}\leqslant(k-1)\beta_1$. To establish (ii), let $\{E_1,E_2,\dots,E_r\}$ be an I_k-partition of E with $r=\chi'_k(G)$, and $\chi'(\langle E_i\rangle)=t_i$. Then $t_i\leqslant k-1$ for each i, and $\chi'(G)\leqslant\Sigma\, t_i\leqslant(k-1)\chi'_k(G)$. □

We now deduce some bounds for χ'_k using (4) and the following results:
If Δ is the maximum degree of G,

$$\Delta\leqslant\chi'\leqslant\Delta+1.\quad[6] \tag{5}$$

If G is bipartite

$$\chi'=\Delta.\quad[4] \tag{6}$$

By (4), (5) and (6), we have for any graph G, if $k\geqslant2$

$$\left\lceil\frac{\Delta}{k-1}\right\rceil\leqslant\chi'_k\leqslant\Delta+1 \tag{7}$$

and if G is bipartite,

$$\chi'_k\leqslant\Delta, \tag{8}$$

Proposition 2. *For any graph G with q edges*

(i) $$\frac{q}{\beta_{1k}}\leqslant\chi'_k\leqslant\frac{q}{k-1},$$

(ii) $$\frac{q}{(k-1)\beta_1}\leqslant\chi'_k\leqslant\left\lceil\frac{q-\beta_{1k}}{k-1}\right\rceil+1.$$

Proof. (i) Let $\{E_1,E_2,\dots,E_r\}$ be an I_k-partition of E with $r=\chi'_k$. Then $q=\Sigma|E_i|\leqslant r\beta_{1k}$, and the lower bound in (i) follows. The upper bound in (i) is trivial. The lower bound in (ii) follows from (i) and (3). To establish the upper bound, let $F\subset E$ be an I_k-set with $|F|=\beta_{1k}$. Clearly, $\chi'_k(G-F)\geqslant\chi'_k-1$. Since $G-F$ has $q-\beta_{1k}$ edges, we have from (i),

$$\chi'_k(G-F)\leqslant\frac{q-\beta_{1k}}{k-1}.$$

Therefore,

$$\chi'_k(G)\leqslant\left\lceil\frac{q-\beta_{1k}}{k-1}\right\rceil+1.\qquad\square$$

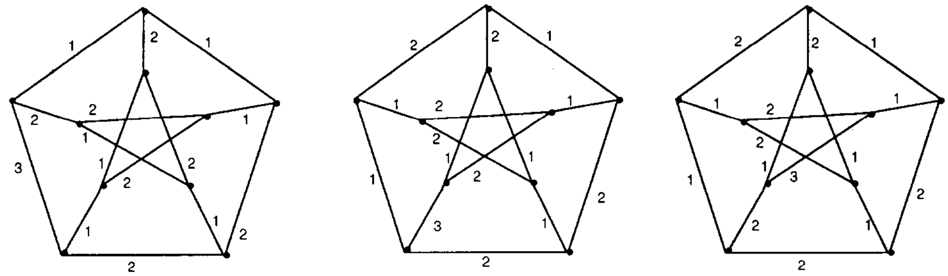

Fig. 3.

(k, n)-Critical Graphs: Let G be a graph with maximum degree Δ. Then G is *chromatic-index critical* (or simply, *Δ-critical*) if (i) G is connected, (ii) $\chi'(G)=\Delta+1$, and (iii) $\chi'(G-e)<\chi'(G)$ for every edge e of G. For details on Δ-critical graphs, see [1] and [7]. We generalize this concept as follows:

Let $k\geqslant 2$ and $n\geqslant 2$ be integers. A graph G is *(k,n)-critical* if (i) G is connected, (ii) $\chi_k'(G)=n$, and (iii) $\chi_k'(G-e)<\chi_k'(G)$ for every edge e of G.

Note that a Δ-critical graph is $(2, \Delta+1)$-critical. For $k\geqslant 3$, the star $K_{1,n}$ is (k,r)-critical, if and only if, $n\equiv 1 \bmod (k-1)$, where $r=\chi_k'(K_{1,n})$. The Peterson graph is $(4,3)$-critical. This can be seen from the $(4,3)$-colorings of the edges as in Fig. 3.

Some elementary properties of (k, n)-critical graphs are as follows.

Proposition 3. *Let G be a (k,n)-critical graph. If $F\subset E$ is an I_k-set, then* (i) $\chi_k'(G-F)=n-1$, (ii) *G contains a (k,r)-critical subgraph for every r satisfying $2\leqslant r\leqslant n$, and* (iii) *if u and v are adjacent vertices* in G, *then* $\deg u+\deg v\geqslant n+1$.

Proof. (i) is trivial.

(ii) For every edge e of G, $\chi_k'(G-e)=n-1$. If the graph $G-e$ is not $(k,n-1)$-critical, we successively remove the edges from $G-e$ until we obtain a graph G' which is $(k,n-1)$-critical. Continuing this process, we can obtain a (k,r)-critical subgraph of G for each r, $2\leqslant r\leqslant n$.

(iii) Clearly there exists a (k, n)-edge coloring of G such that $\{e\}$ is a color class. Let $\{e\}, E_2, E_3, \ldots, E_n$ be the color classes in such an edge coloring. The edge e should be adjacent to at least one edge in each color class E_i, $2\leqslant i\leqslant n$. This implies $(\deg u-1)+(\deg v-1)\geqslant n-1$.

A graph G is *(k, n)-vertex critical* if $\chi_k(G)=n$ and $\chi_k(G-v)=n-1$ for all $v\in V$. We deduce our next result using a known result.

Proposition 4 (Sampathkumar [5]). *Let G be a (k, n)-vertex critical graph, $n\geqslant 2$. Then* (i) *G is $(n-1)$-edge connected, and* (ii) *$\delta(G)\geqslant n-1$, where $\delta(G)$ is the minimum degree of G.*

Clearly, $\delta(L(G)) = \min\{\deg u + \deg v : uv \in E\} - 2$. Since $\chi'_k(G) = \chi_k(L(G))$, and G is (k, n)-critical $\Leftrightarrow L(G)$ is (k, n)-vertex critical, we deduce the following proposition from Proposition 4:

Proposition 5. *Let G be a (k, n)-critical graph, $n \geqslant 2$. Then* (i) *$L(G)$ is $(n-1)$-edge connected.*

Corollary 5.1. *Let G be a Δ-critical graph. Then $L(G)$ is Δ-edge connected.*

We now present an upper bound on the number of edges in a (k, n)-critical graph.

Proposition 6. *Let $d_1, d_2, \ldots, d_p$ be the degree sequence of a (p, q) graph G. If G is (k, n)-critical then $q \leqslant \Sigma d_i^2/(n+1)$.*

Proof. The number of edges in the line graph $L(G)$ of G is given by $q_L = -q + \frac{1}{2}\Sigma d_i^2$. Let $d'_1, d'_2, \ldots, d'_q$ be the degree sequences of $L(G)$. By (ii) of Proposition 5, $d'_1 \geqslant \delta(L(G)) \geqslant n-1$ for each i. Hence,

$$2q_L = \sum_{i=1}^{q} d'_i \geqslant q(n-1), \quad \text{and} \quad q \leqslant \frac{-2q + \Sigma d_i^2}{n+k-3}$$

and the result follows.

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