A NOTE ON THREE ALLIED PROBLEMS OF RAMANUJAN

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1. S. Ramanujan had set the following three problems:—

Show that

\[(6a^2 - 4ab + 4b^3)^3 = (3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3\]
\[+ (5a^2 - 5ab - 3b^2)^3\]

and find other quadratic expressions satisfying similar relations.\(^1\) (R. I)

Solve, in integers, \(x^3 + y^3 + z^3 = 1\) and deduce

(i) \(6^3 + 8^3 = 9^3 - 1\)
(ii) \(9^3 + 10^3 = 12^3 + 1\)
(iii) \(135^3 + 138^3 = 172^3 - 1\)

(iii) \(791^3 + 812^3 = 1010^3 - 1\)
(iv) \(11161^3 + 11468^3 = 14258^3 + 1\)
(v) \(65601^3 + 67402^3 = 83802^3 + 1^2\)

(R. II)

Solve, in integers, \(x^3 + y^3 + z^3 = \mu^6\) and deduce that

(i) \(6^3 - 5^3 - 3^3 = 2^6\)
(ii) \(8^3 + 6^3 + 1^3 = 3^6\)
(iii) \(12^3 - 10^3 + 1^3 = 3^6\)

(iv) \(46^3 - 37^3 - 3^3 = 6^6\)
(v) \(174^3 + 133^3 - 45^3 = 14^6\)
(vi) \(1188^3 - 509^3 - 3^3 = 34^6\) (R. III)

(R. I) was solved by S. Narayan\(^4\) who, by replacing 6, 5, 4, 3 in the coefficients above by \(l, m, n, p\) obtained \(l = \lambda (\lambda^3 + 1), m = 2 \lambda^3 - 1, n = \lambda (\lambda^3 - 2)\) and \(p = \lambda^3 + 1\), which set, with \(\lambda = 2\), reduces to (R. I). His values are not completely general and are just similar to Vieta’s formulæ.\(^5\)

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\(^1\) A more general investigation of this and similar problems will be found in the article on “Types of solutions of \(x^3 + y^3 + z^3 = 1\) in integers,” vide J.I.M.S., New series, 4, 2.

\(^2\) Vide J.I.M.S., 5, old series, p. 39, Q. 441.

\(^3\) Ibid., 7, old series, p. 160, Q. 681.

\(^4\) Ibid., 7, old series, Q. 661

Vieta gave \( x = B (B^3 - 2D^3), \ y = D (2B^3 - D^3), \ z = B (B^3 + D^3) \) and \( w = -D (B^3 + D^3) \) as satisfying \( x^3 + y^3 + z^3 + w^3 \). If, in this, \( D = 1, \ B = \frac{1}{\lambda} \), we derive Narayan’s set. A very particular solution of (R. II) was given by N. B. Mitra\(^8\) who gave \( x = 3a (1 - 3a^2), \ y = 9a^4, \ z = 1 - 9a^3 \) as a solution of \( x^3 + y^3 + z^3 = 1 \) and for \( a = \pm 1 \), verified (i) and (ii) in (R. II). Mitra also solved (R. III) giving two distinct solutions; yet, he left (vi) in (R. III) unverified. In the present note, an attempt is made to give an easy method of completely solving all the three problems, verifying every numerical example.

§ 2. Consider \( x^3 + y^3 = k^3 (z^3 + w^3) \). Take \( k = 2 \). Put \( x + y = p, \ x - y = q, \ z + w = r, \ z - w = s \) (all integers). We get \( p (p^2 + 3q^2) = 8r (r^2 + 3s^2) \). If \( p = 8r \), then \( q^2 + 21r^2 = s^2 \). One evident solution of this is \( q = a^2 - 21b^2, \ r = 2ab, \ s = a^2 + 21b^2 \). Hence, on substitution and reduction,

\[
(a^2 + 16ab - 21b^2)^3 + (-a^2 + 16ab + 21b^2)^3 = (2a^2 + 4ab + 42b^2)^3
\]

\[
+ (-2a^2 + 4ab - 42b^2)^3
\]

(I)

It may be remarked here that, in (I), if \( a = d, \) and \( b = 1 \) we get Young’s identity\(^7\); multiplying both sides of (I) by \(-27\), putting \( \beta \) for \( 3a \) and \( a \) for \( 9b \), we get Gerardin’s identity.\(^8\) Replace \( a \) in I by \( 3(a + b) \), and \( b \) by \( (a - b) \), you obtain (R. I).

§ 3. Now, \( x = a^2 + 16ab - 21b^2 \) in (I), i.e., \( x = (a + 8b)^2 - 85b^2 \). Choose \( a \) and \( b \) in such a manner that \( \frac{a + 8b}{b} \) is numerically equal to a convergent of the C. F. for \( \sqrt{85} \), viz., \( 9 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{18}}}}} \) so that, from the first period itself, we get

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\(^6\) Vide J.I.M.S., 13, old series, pp. 15 and 17.

\(^7\) Vide Dickson: History of the Theory of Numbers, 2, p. 559

\(^8\) Ibid., 2, p. 559.
of these, (II·1), (II·2), (II·7), (II·8) and (II·9) are (ii) (iii), (iv), (v) and (vi) in (R·II). (i) in (R·I) is got by combining (II·1) with Euler's identity in the form $6^3 + 8^3 = 12^3 - 10^3$. (II·10) is extra.

§ 4. Instead of trying to reduce any one of $x$, $y$, $z$, $w$ to unity, let us try to make $y$, $z$, $w$ simultaneously multiples of $x$. Assume

$$-a^2 + 16 ab + 21 b^2 = m (a^2 + 16 ab - 21 b^2).$$

This gives

$$\frac{a}{b} = -\frac{8 (m - 1)}{m + 1},$$

where

$$85 m^2 - 86 m + 85 = n^2.$$

Take

$$a = -8 (m - 1) + n, \quad b = m + 1.$$

Then $x = 32 a, \quad y = 32 am, \quad z = 2 a (n + m + 1), \quad w = 2 a (-n + m + 1)$

yielding

$$1^3 + m^3 = \left(\frac{n + m + 1}{8}\right)^3 + \left(\frac{-n + m + 1}{8}\right)^3.$$

Put $m + 1 = 8 \lambda$ and $n = 8 \mu$, we get

$$1 + (8 \lambda - 1)^3 = (\lambda + \mu)^3 + (\lambda - \mu)^3 \quad \{ \text{III} \}$$

where

$$85 \lambda^2 - 32 \lambda + 4 = \mu^2.$$
For

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$-11$</td>
<td>II.1 or R. II.2</td>
</tr>
<tr>
<td>$17$</td>
<td>$-155$</td>
<td>II.2 or R. II.3</td>
</tr>
<tr>
<td>$99$</td>
<td>$911$</td>
<td>II.7 or R. II.4</td>
</tr>
<tr>
<td>$-1395$</td>
<td>$12863$</td>
<td>II.8 or R. II.5</td>
</tr>
<tr>
<td>$-8200$</td>
<td>$-75602$</td>
<td>II.9 or R. II.6</td>
</tr>
<tr>
<td>$115784$</td>
<td>$-1067474$</td>
<td>II.10</td>
</tr>
</tbody>
</table>

§ 5. Similarly, consider $x^3 + y^3 = 3^3(z^3 + w^3)$.

If, as before, $x = p + q$, $y = p - q$, $z = r + s$, $w = r - s$, and $p$ is taken = $9r$, then $26r^2 = 3s^2 - q^2$. Let $s = 3r + \lambda$. Then $q^2 = (r + 9\lambda)^2 - 78\lambda^2$.

One evident solution is $r + 9\lambda = a^2 + 78b^2$

$$\lambda = 2ab$$

and

$$q = a^2 - 78b^2$$

with

$$r = a^2 - 18ab + 78b^2,$$

$$p = 9(a^2 - 18ab + 78b^2),$$

and

$$s = 3a^2 - 52ab + 234b^2,$$

leading to corresponding expressions for $x$, $y$, $z$, $w$. Therein, on replacing $a$ by $(8a + 2b)$ and $b$ by $a$, we derive

$$(a^2 + 17ab - 8b^2)^3 + (8a^2 + ab - 10b^2)^3 = 3^3[(3a^2 - ab + 2b^2)^3 - (2a^2 - 3ab + 4b^2)^3]$$

(IV)

§ 6. Now, $a^2 + 17ab - 8b^2 = \left(a + 17\frac{b}{2}\right)^2 - 321\left(\frac{b}{2}\right)^2$, so that if

$$a + 17\frac{b}{2}$$

$$\pm\frac{b}{2}$$

give a convergent of the C. F. for $\sqrt{321}$, viz., $17 + \frac{1}{1 + \frac{1}{10 + \frac{1}{1 + \frac{1}{34}}}}$, we get, from the first period,
§ 7. Here, take \( y = mx \). Then \( \frac{a}{b} = \frac{-17m + 1 \pm n}{2 (m - 8)} \) where \( n^2 = 321m^2 - 330m + 321 \).

Putting \( a = -17m + 1 + n \),
\[
\frac{b}{2} = 2 (m - 8), \quad \text{and} \quad k = -5n + 85m - 37,
\]
we get \( x = 54k, y = 54mk, z = k(-n + 3m + 3) \) and \( w = k(-n - 3m - 3) \), so that, on further supposing \( m = 9\lambda - 1 \) and \( n = -9\mu \), we have
\[
1^3 + (9\lambda - 1)^3 = \left(\frac{3\lambda + \mu}{2}\right)^3 + \left(\frac{3\lambda - \mu}{2}\right)^3
\]
where \( 321\lambda^2 - 108\lambda + 12 = \mu^2 \).

Here, suitable values of \( \lambda \) and \( \mu \) lead to the results (ii), (iii), (iv), (vii), (viii) under (IV).

§ 8. Lastly, take \( x^3 + y^3 = 4^3(z^3 + w^3) \).

Proceeding as before, \( p(p^2 + 3q^2) = 64r(r^2 + 3s^2) \).

Let \( p = 64r \). Then \( 1365r^2 + q^2 = s^2 \).

One evident solution is \( q = a^2 - 1365b^2 \)
\[
r = 2ab
\]
and \( \therefore \)
\[
s = a^2 + 1365b^2
\]
\[
p = 128ab.
\]
So that we obtain
\[ x = a^2 + 128 ab - 1365 b^2 \]
\[ y = -a^2 + 128 ab + 1365 b^2 \]
\[ z = a^2 + 2 ab + 1365 b^2 \]
\[ w = -a^2 + 2 ab - 1365 b^2 \]
giving
\[ x^3 + y^3 = 4^3 (z^3 + w^3). \]
Replacing \( a \) by 21 \( a \) and reducing,
\[ (21 a^2 + 128 ab - 65 b^2)^3 + (-21 a^2 + 128 ab + 65 b^2)^3 \]
\[ = 4^3 [(21 a^2 + 2 ab + 65 b^2)^3 - (21 a^2 - 2 ab + 65 b^2)^3]. \] (VI)

Here, if we make one of the expressions like 21 \( a^2 + 2 ab + 65 b^2 \) a perfect square, we can verify all the results (R. III) (i) to (vi). For example, with \( a = 13, \ b = -2 \), we obtain the sixth result which was not verified by Mitra, \( viz., -509^3 - 3^3 + 1188^3 = (34)^6. \)

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9 This note was the subject-matter of two lectures delivered before the Mathematical Societies of the Central College, Bangalore, and the Presidency College, Madras.