Thermal Instability in the Presence of Hall-Current

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We consider the effect of Hall-current on the thermal instability which was studied by Field (1965) with a view to explaining the existence of non self-gravitating astronomical objects. We find that Hall-current does not change the stability criteria but introduces additional modes of propagation which do not give rise to any new criteria. However, it affects the growth rates of other modes.

Key words: thermal instability — Hall-current

1. Introduction

As pointed out by Field (1965) though most astronomical objects owe their existence to self-gravitation, there is a class of objects including solar prominences, interstellar clouds and condensation in planetary nebulae whose existence cannot be explained due to self-gravitation. Consequently, he studied the possibility of a condensation process not involving gravitation but as a consequence of instability in thermal equilibrium. This problem has been studied earlier by many authors whose work has been critically reviewed by Field. Field investigated the problem completely in detail and obtained instability criteria. He further studied the effects of rotation and magnetic field on the instability. Recently Hunter (1966) extended this problem to include gravitational effects also.

As pointed out by Cowling (1957) and Lighthill (1960) in a fully ionized low density medium the Larmor frequency of electrons is large compared to their collision frequency and hence the effect of Hall-current becomes important. In this note we have taken the Hall-current into account in the above problem employing the generalized Ohm’s law given by Spitzer (1962).

Nomenclature

$\varepsilon_0, p_0, T_0$: Steady state density, pressure and temperature. $c$: Speed of light in vacuum.
$\mathcal{L}(\phi, T_0)$: Generalized heat loss function. $\omega_p$: Electron plasma frequency.
$\gamma$: Ratio of specific heats of the medium. $K$: Thermal Conductivity.
$-e$: Charge on an electron. $k$: Wave number.
$\eta$: Resistivity. $c_s$: Speed of sound in the undisturbed medium.
$n$: Number density of electrons. $R$: Gas Constant.

2. Linearized Equations and Dispersion Relation

Let $\mathbf{v}, \phi, p_1, T$, and $\mathbf{h}$ denote small perturbations in velocity, density, pressure, temperature and magnetic field respectively. The linearized basic equations are:

$$\frac{\partial \phi_1}{\partial t} + \varepsilon_0 \mathbf{V} \cdot \mathbf{v} = 0$$

$$\varepsilon_0 \left[ \frac{\partial \mathbf{v}}{\partial t} + 2 \mathbf{V} \times \mathbf{v} \right] = -\nabla p_1 + \frac{1}{4\pi} (\nabla \times \mathbf{h}) \times \mathbf{h}$$

$$\frac{1}{\gamma - 1} \left[ \frac{\partial p_1}{\partial t} - \frac{\gamma p_0}{\varepsilon_0} \frac{\partial \phi_1}{\partial t} \right] + \phi_0 [\mathcal{L}_\phi \phi_1 + \mathcal{L}_T T_1] - \kappa \Delta T_1 = 0$$

$$p_1/p_0 = (T_1/T_0) + (\phi_1/\phi_0)$$

$$\frac{\partial \mathbf{h}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{h}) + \eta \Delta \mathbf{h} - \frac{c}{4\pi n e} \nabla \times [\mathbf{V} \times \mathbf{h}] \times \mathbf{h} + \frac{e^2}{\omega_p^2} \frac{\partial}{\partial t} (\Delta \mathbf{h})$$.
where $\mathbf{\Omega}$ and $\mathbf{H}_0$ are uniform rotation and uniform magnetic field present in the medium and $\mathbf{L}_q = \frac{\partial \mathbf{L}}{\partial \theta}$, $\mathbf{L}_T = \frac{\partial \mathbf{L}}{\partial \theta_T}$ are evaluated for the steady state. The last equation has been obtained by using the generalized Ohm’s law and Maxwell’s equations (in e.m.u.).

Choosing the coordinate system such that $\mathbf{H}_0 = (H_z, 0, H_z)$ while $\mathbf{\Omega} = (\Omega'_x, \Omega'_y, \Omega'_z)$ we seek a solution of (2.1) assuming perturbed quantities are proportional to $exp(\sigma t + ikz)$. The condition of nontriviality of the solution leads to the dispersion relation:

$$
(w - \sigma_z + \sigma'_T)(w - \sigma_e)\left\{[(w - \sigma_e)(w^3 + \Omega^2_H) + \Omega^2_A w]\left[4\Omega^2_T(w^3 + \Omega^2_H) + 4\Omega_A\Omega_B\Omega_C\Omega_H\right]ight. \\
\times \left. \left[w^3 + \Omega^2_H - \Omega^2_A\Omega^2_B(w^3 - \Omega^2_H)\right] - 2\Omega_A\Omega_B w[2\Omega_A(w^3 + \Omega^2_H) + \Omega_A\Omega_B\Omega_H]\right\}
$$

$$
\times \left[2\Omega_A(w^3 + \Omega^2_H) + \Omega^2_A\Omega_H\right] + \left[-\left(\frac{1}{\gamma} - \sigma_e \right) + \frac{\sigma_e}{\gamma}\right](w^3 + \Omega^2_H) + (w - \sigma_e + \sigma'_T)
$$

$$
\times \left\{[(w - \sigma_e)^2 + \frac{1}{\gamma}](w^3 + \Omega^2_H) + \Omega^2_B w(w - \sigma_e)\right\} \left\{[(w^3 + \Omega^2_H)(w - \sigma_e) + \Omega^2_A w]^2 \right. \\
\left. + 2\Omega_A(w^3 + \Omega^2_H) + \Omega^2_A\Omega_H\right\} = 0
$$

(2.2)

where,

\[ w = (\sigma + \eta k^2/\lambda)/k c_s, \sigma_e = \eta k/l c_s, \sigma_q = k_0/k, \sigma_T = k_T/k, \sigma_K = k/K, \sigma'_T = \sigma_T + \sigma_K, \]

\[ \Omega_H = c H/\lambda, 4\pi n e c_s, \Omega^2_{A,B} = H^2/4\pi n^2 c_s^2, \lambda = \Omega/k c_s, \lambda = 1 + k^2 c^2/\omega_p^2, \]

\[ k_0 = L^2_0/(\gamma - 1) g_0/R c_s T_0, k_T = (\gamma - 1) L^2_0/R c_s, k_K = R c_s/\gamma K (\gamma - 1). \]

3. Discussion of the Dispersion Relation and Conclusion

Case 1: Transverse Magnetic Field

In this case, in the presence of purely transverse rotation the dispersion relation (2.2) reduces to

$$
w^3 + \sigma'_T w^2 + (1 + 4\Omega^2_H + \Omega^2_A) w + \left[\sigma'_T (4\Omega^2_H + \Omega^2_A) + \frac{\sigma'_e - \sigma_e}{\gamma}\right] = 0
$$

(2.3)

while in the case of longitudinal rotation, (2.2) reduces to,

$$
w^3 + 4\Omega^2_A = 0,
$$

(2.4)

and

$$
w^3 + \sigma'_B w^2 + (1 + \Omega^2_B) w + \left(\frac{1}{\gamma} + \Omega^2_B\right) \sigma'_T - \frac{\sigma_e}{\gamma} = 0.
$$

(2.5)

The two cubic equations are the same except that in the former the effect of transverse rotation is present. If the discriminant of the cubic is positive, the cubic has a real root and a pair of complex roots; the mode corresponding to the real root is stable or unstable depending on whether $k \gtrless k^*$ where

$$
k^* = k^*_T \left[k^*_T \left(k^*_T V^2_T + 4\Omega^2_A + \frac{1}{\gamma}\right)\right] = k^*_T \left[1 + (\mathbf{\sigma} k^*_T k^*_K V^2_T/\gamma \omega_p^2) + 0(1/\omega_p^2)\right]
$$

(2.6)

which shows that the critical wavelength is enhanced due to the presence of transverse magnetic field and rotation and the electron inertia through the plasma frequency.

The modes corresponding to the pair of complex roots of the cubic are stable or unstable depending on whether

$$
\sigma'_T (1 - \gamma)/\gamma - \sigma'_e/\gamma \gtrless 0.
$$

(2.7)

It is interesting to note that this criterion is independent of magnetic field, rotation and the electron plasma frequency. If the discriminant of the cubic is negative, the system can be classified as stable or unstable depending on the signs of $\sigma'_T$ and $\sigma'_T (\Omega^2_B + 4\Omega^2_A + \frac{1}{\gamma}) - \frac{\sigma_e}{\gamma}$. This does not differ from the classification of Field.
Case II: Longitudinal Magnetic Field

Here the dispersion relation reduces to \( w^2 + \Omega_H^2 = 0 \), which simply gives two stable modes similar to Alfvén type of waves, besides a ninth degree polynomial equation.

Equation (2.2) is of eleventh degree and in the absence of Hall-current reduces to seventh degree and consequently Hall-current introduces four additional modes, whose frequencies tend to zero as \( \Omega_H \to 0 \). In this limit the frequencies are approximately given by:

\[
- \frac{\Omega_H^2}{4 (\sigma_T - \sigma_p)} \pm \frac{i \Omega_H}{\sqrt{2}} , \quad - \frac{3 \gamma \Omega_H^2}{4 (\sigma_T - \sigma_p)} \pm i \sqrt{2} \Omega_H
\]

and it is clear that these modes are stable or unstable depending on whether:

\[
\sigma_T - \sigma_p \gtrless 0.
\]

Thus, the criterion is unaffected by Hall-current. However, it affects the growth rates of other unstable modes.

Finally, we have solved the dispersion relation (2.2) numerically when both longitudinal and transverse magnetic fields are present for both finite and infinite conductivities taking the following numerical values for the various parameters.

\[
\gamma = 1.6667 , \quad c_s = 1.5 \times 10^6 \text{ cm/s} , \quad c = 3 \times 10^{10} \text{ cm/s} , \quad \frac{\lambda c_s \Omega_H}{k} = 0.68 \times 10^{14} / \text{s} \quad \sigma_p^2 = 0.32 \times 10^{10} / \text{s}^2 ,
\]

\[
\Omega_A = 2 , \quad \Omega_B = 3 , \quad \Omega'_A = \Omega'_B = 10^{-15} / \text{s} , \quad \Omega_z = 0 , \quad k_R = 0.2 \times 10^{-7} / \text{cm}.
\]

![Fig. 1](image)

<table>
<thead>
<tr>
<th>(k_q(\text{cm}))</th>
<th>(k_T(\text{cm}))</th>
<th>(k_q(\text{cm}))</th>
<th>(k_T(\text{cm}))</th>
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<td>10^{-8}</td>
<td>0.2</td>
</tr>
<tr>
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<tr>
<td>(C)</td>
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<td>0.2</td>
</tr>
<tr>
<td>(D)</td>
<td>5</td>
<td>10^{-8}</td>
<td>0.2</td>
</tr>
<tr>
<td>(E)</td>
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<td>10^{-7}</td>
<td>0.2</td>
</tr>
<tr>
<td>(F)</td>
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</tr>
<tr>
<td>(G)</td>
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<td>10^{-8}</td>
<td>0.8</td>
</tr>
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</table>

We have solved for different sets of values of \(k_q\) and \(k_T\) over a wide range of values of \(k\).

Figure 1 shows the distribution of the maximum growth rate with the wave number for different sets of values of \(k_q\) and \(k_T\). These curves always show an extreme value for some particular values of \(k\).

The table shows the maximum growth rates for a particular set of values of \(k_q\) and \(k_T\) when the conductivity is finite and infinite. We find that finite conductivity affects the growth rates slightly. Because of the large linear dimensions, the dimensionless parameter involving conductivity is very small and this explains the fact that in the physical situation we are considering, finite conductivity does not affect the results qualitatively.
Table

<table>
<thead>
<tr>
<th>k/cm</th>
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<tr>
<td></td>
<td>(i)</td>
<td>(ii)</td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>$0.2715$</td>
<td>$10^{-10}$ $0.3396$ $10^{-10}$</td>
</tr>
<tr>
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<td>$0.7016$</td>
<td>$10^{-9}$ $0.7016$ $10^{-9}$</td>
</tr>
<tr>
<td>$10^{-11}$</td>
<td>$0.9128$</td>
<td>$10^{-11}$ $0.9128$ $10^{-11}$</td>
</tr>
<tr>
<td>$10^{-16}$</td>
<td>$0.8750$</td>
<td>$10^{-16}$ $0.8750$ $10^{-16}$</td>
</tr>
</tbody>
</table>

Table showing the maximum growth rates: (i) infinite conductivity, (ii) finite conductivity corresponding to $\mu_0 = 10^{-9}/cm$, $\mu_r = 0.2$ $10^{-4}/cm$, $\mu_k = 0.2$ $10^{-7}/cm$

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