A NOTE ON PONCELET'S PROBLEM

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Introduction

If $P_1, P_2, P_3, \ldots$ are points on a conic $S_1$ which are such that $P_rP_{r+1}$ ($r = 1, 2, 3, \ldots$) touch another conic $S_2$, then it is well known that $P_1P_m$ (for every point $P_1$ on $S_1$) touches a conic of the pencil formed by them, say $S_m = S_1 + \lambda_m S_2$, where $\lambda_m$ is a rational function of the invariants of the two conics. The calculation of $\lambda_m$ becomes very complicated for higher values of $m$. A recurrence formula for their calculation can be obtained by using elliptic functions. Here I obtain a recurrence formula by a simple geometric method.

1. At first I shall explain the notation adopted and prove briefly some preliminary results. Let $\Sigma_1, \Sigma_2$ be the reciprocal forms of $S_1, S_2$. Let the discriminant of $S_1 + \lambda S_2$ be $\Delta_1 + \lambda \Delta_{12} + \lambda^2 \Delta_{21} + \lambda^3 \Delta_2$. Let the reciprocal form of $S_1 + \lambda S_2$ be $\Sigma_1 + \lambda \Sigma_{12} + \lambda^2 \Sigma_2$, and the reciprocal form of $\Sigma_1 + \lambda \Sigma_2$ be $\Delta_1 S_1 + \lambda S_{12} + \lambda^2 \Delta_2 S_2$. $\Sigma_{12} = 0, S_{12} = 0$ are the $\phi$-envelope and F-conic of the two conics. The reciprocal form of $k_1 \Sigma_1 + k_2 \Sigma_2 + k_3 \Sigma_3$ is a quadratic form in $k_1, k_2, k_3$. It must therefore be equal to $k_1^2 \Delta_1 S_1 + k_2^2 \Delta_2 S_2 + k_3^2 \Delta_3 S_3 + k_1 k_2 \Delta_{12} S_1 S_2 + k_1 k_3 \Delta_{13} S_1 S_3 + k_2 k_3 \Delta_{23} S_2 S_3$ where $\Delta_{ij}$ is the F-conic form of $\Sigma_i, \Sigma_j$. Therefore the reciprocal form of $\Sigma_1 + \lambda \Sigma_{12} + \lambda^2 \Sigma_2$ must be equal to $\Delta_1 S_1 + \lambda^2 \Delta_{12} S_{12} + \lambda^3 \Delta_{21} S_{11} + \lambda^4 \Delta_2 S_{22}$ where $\Delta_{ij}$ is the reciprocal form of $S_{ij}$, and $S_{ij}$ is the F-conic form of $\Sigma_i, \Sigma_j, (i = 1, 2)$. But the reciprocal form of $\Sigma_1 + \lambda \Sigma_{12} + \lambda^2 \Sigma_2$ must also be equal to $(\Delta_1 + \lambda \Delta_{12} + \lambda^2 \Delta_{21} + \lambda^3 \Delta_2) (S_1 + \lambda S_2)$. Therefore

\[
(\Delta_1 + \lambda \Delta_{12} + \lambda^2 \Delta_{21} + \lambda^3 \Delta_2) (S_1 + \lambda S_2) = \Delta_1 S_1 + \lambda S_{12} + \lambda^2 (P_{12} + S_{12}) + \lambda^3 S_{11} + \lambda^4 \Delta_2 S_2
\]

which gives

\[
S_{1,12} = \Delta_{12} S_1 + \Delta_1 S_2 \quad (1.1)
\]

\[
S_{2,12} = \Delta_2 S_1 + \Delta_{21} S_2 \quad (1.2)
\]

\[
P_{12} + S_{12} = \Delta_{21} S_1 + \Delta_{12} S_2 \quad (1.3)
\]

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The F-conic form of $\Sigma_1$ and $a\Sigma_2 + \beta \Sigma_3$ is easily seen to be $aS_{12} + \beta S_{13}$, and the F-conic form of $\Sigma_1$ and $\Sigma_1$ is seen to be $2 \Delta_1 S_1$. Therefore the F-conic form of $S_1$ and $S_1 + \alpha S_2$, i.e., the F-conic form of $\Sigma_1$ and $\Sigma_1 + a\Sigma_2 + a^2 \Sigma_2$ is

$$2 \Delta_1 S_1 + aS_{1,12} + a^2 \Delta_2 S_2. \tag{1\cdot4}$$

We know that the equation of common tangents of $\Sigma_1$ and $\Sigma_2$ is

$$4 \Delta_1 \Delta_2 S_1 S_2 - S_{12}^2 = 0.$$ 

Therefore the equation of common tangents of $S_2$ and $S_1 + \beta S_2$, i.e., of $\Sigma_2$ and $\Sigma_1 + \beta \Sigma_1 + \beta^2 \Sigma_2$ is

$$4 \Delta_2 (\Delta_1 + \beta \Delta_{1,2} + \beta^2 \Delta_{21} + \beta^3 \Delta_2) S_2 (S_1 + \beta S_2) - (S_{12} + \beta S_{2,12} + \beta^2 \cdot 2 \Delta_2 S_2)^2 = 0. \tag{1\cdot5}$$

2. Let $Q_1$ be the point of contact with $S_1$ of a common tangent of $S_1$ and $S_2$. Let $Q_2, Q_3, Q_4, \ldots$ be points on $S_1$ such that $Q_r$, $Q_{r+1}$ $(r = 1, 2, 3, \ldots)$ touch $S_2$. If we take $P_2$ to lie at $Q_h$, then $P_1, P_2, P_3, \ldots, P_{2k}$ (where $P_r, P_{r+1}, r = 1, 2, \ldots, 2k - 1$ touch $S_2$) are seen to be the points $Q_{k}, Q_{k-1}, \ldots, Q_2, Q_1, Q_0, Q_2, \ldots, Q_k$ respectively.

It follows therefore that

$$(2\cdot1)$$ the tangent at $Q_k$ to $S_1$ (a particular $P_1P_{2k}$) is a tangent to $S_{2k}$;

$$(2\cdot2)$$ the line $Q_2Q_{k-1}$ (a particular $P_1P_2$ and $P_1P_{2k-1}$) is a common tangent to $S_2$ and $S_{2k-1}$;

$$(2\cdot3)$$ the tangent at $Q_{k-1}$ to $S_1$ is a tangent to $S_{2k-2}$.

From these we deduce that $Q_k$ lies on the F-conic of $S_1$ and $S_{2k}$; it lies on $Q_kQ_{k-1}$ which is a tangent to $S_2$ and $S_{2k-1}$; it also lies on $Q_kQ_{k+1}$ which is a tangent to $S_2$ and $S_{2k+1}$. Also $Q_kQ_{k-1}$, a common tangent of $S_2$ and $S_{2k-1}$ passes through $Q_k$ and $Q_{k-1}$, i.e., a point of intersection of $S_1$ and the F-conic of $S_1$ on $S_{2k}$ and a point of intersection of $S_1$ and the F-conic of $S_1$ and $S_{2k-1}$. We obtain therefore that

$$(2\cdot4)$$ the F-conic of $S_1$ and $S_{2k}$ intersects $S_1$ in points which lie on the common tangents of $S_2$ and $S_l$, where $l$ may be $2k+1$ or $2k-1$.

$$(2\cdot5)$$ the common tangents of $S_2$ and $S_{2k-1}$ pass through the points of intersection of $S_1$ and the F-conic of $S_1$ and $S_l$, where $l$ may be $2k$ or $2k-2$.

3. The F-conic of $S_1$ and $S_1 + \alpha S_2$ is

$$(2 \Delta_1 + a \Delta_{13}) S_1 + \alpha \Delta_1 S_2 + a^2 S_{12} = 0 \tag{3\cdot1}$$
from (1·4) and (1·1). The equation of common tangents of $S_2$ and $S_1 + \beta S_2$ is
\[
4 \, \Delta_2 \, (\Delta_1 + \beta \Delta_{12} + \beta^2 \Delta_{21} + \beta^3 \Delta_2) \, (S_2) \, (S_1 + \beta S_2)
- (S_{12} + \beta \Delta_2 S_1 + \beta \Delta_{21} S_2 + 2\beta^2 \Delta_2 S_2)^2 = 0
\] (3·2)

from (1·5) and (1·2).

Points satisfying $(2 \Delta_1 + \alpha \Delta_{12}) \, S_1 + \alpha \Delta_1 S_2 + \alpha^2 S_{12} = 0$ and $S_1 = 0$ also satisfy
\[
4 \, \Delta_2 \, (\Delta_1 + \beta \Delta_{12} + \beta^2 \Delta_{21} + \beta^3 \Delta_2) \, S_2 \, (S_1 + \beta S_2)
- (S_{12} + \beta \Delta_2 S_1 + \beta \Delta_{21} S_2 + 2\beta^2 \Delta_2 S_2)^2 = 0
\]

if the co-ordinates of the points satisfy
\[
\alpha \Delta_2 S_2 + \alpha^2 S_{12} = 0 \quad (3·3)
\]
and
\[
(4\beta \Delta_1 \Delta_2 + 4\beta^2 \Delta_{12} \Delta_2 - \beta^2 \Delta_{21}^2) \, S_2^2 - S_{12}^2
- 2 \, S_{12} S_2 \, (\beta \Delta_{21} + 2 \beta^2 \Delta_2) = 0 \quad (3·4)
\]

Substituting in (3·4) for $S_2^2$ the value at the points from (3·3) we get
\[
\alpha^2 \beta \left( 4 \Delta_1 \Delta_2 + \beta \left( 4 \Delta_{12} \Delta_2 - \Delta_{21}^2 \right) \right)
+ 2 \alpha \beta \left( \Delta_1 \Delta_{21} + 2 \beta \Delta_1 \Delta_2 \right) - \Delta_{12}^2 = 0. \quad (3·6)
\]

Now from the preceding results we obtain the desired recurrence formula, viz., if $\alpha = \lambda_{2k}$ the resulting quadratic in $\beta$ must have its roots equal to $\lambda_{2k-1}$ and $\lambda_{2k+1};$ if $\beta = \lambda_{2k+1}$ the resulting quadratic in $\alpha$ must have its roots equal to $\lambda_{2k}$ and $\lambda_{2k+2}$. Obviously $\lambda_1 = 0, \frac{1}{\lambda_2} = 0$.

4. We calculate $\lambda_3, \lambda_4, \lambda_5$ to illustrate the procedure. Putting $\alpha = \lambda_2$ in (3·6) the two roots in $\beta$ must be $\lambda_1$ and $\lambda_3$. The quadratic in $\beta$ is $\beta \left( 4 \Delta_1 \Delta_2 + 4 \beta \Delta_{12} \Delta_2 - \beta \Delta_{21}^2 \right) = 0$. The two roots are 0 and
\[
\frac{4 \Delta_1 \Delta_2}{\Delta_{21}^2 - 4 \Delta_{12} \Delta_2}.
\]

Also $\lambda_1 = 0, \therefore \lambda_3 = \frac{4 \Delta_1 \Delta_2}{\Delta_{21}^2 - 4 \Delta_{12} \Delta_2}.
\]

Now put $\beta = \lambda_3 = \frac{4 \Delta_1 \Delta_2}{\Delta_{21}^2 - 4 \Delta_{12} \Delta_2}$ in (3·6).
The resulting equation in $\alpha$ is

$$\frac{1}{\alpha} \frac{8 \Delta_1 \Delta_2}{\Delta_{21}^2 - 4 \Delta_2 \Delta_{12}} \left\{ \frac{\Delta_{21}^2 - 4 \Delta_2 \Delta_{12} \Delta_{21} + 8 \Delta_1 \Delta_{12}^2}{\Delta_{21}^2 - 4 \Delta_2 \Delta_{12}} \right\} \cdot \Delta_1 - \frac{\Delta_{12}^2}{\alpha^2} = 0$$

which is satisfied if $\frac{1}{\alpha} = 0$ and $\alpha = \frac{(\Delta_{21}^2 - 4 \Delta_2 \Delta_{12})}{8 \Delta_2 (\Delta_{21}^2 - 4 \Delta_2 \Delta_{12} \Delta_{21} + 8 \Delta_1 \Delta_{12}^2)}$.

Therefore $\lambda_4 = \frac{(\Delta_{21}^2 - 4 \Delta_2 \Delta_{12})^2}{8 \Delta_2 (\Delta_{21}^2 - 4 \Delta_2 \Delta_{12} \Delta_{21} + 8 \Delta_1 \Delta_{12}^2)}$.

Putting $\alpha = \lambda_4$ in (3.6) and dividing the resultant quadratic by the factor $\beta - \lambda_3$, the quotient $= 0$ gives the root which is $= \lambda_5$. 