NOTES ON SUMMABILITY

I. An Equivalence Theorem in a General Field of Summability

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§ 1. Introduction

Theorems such as those of Hardy and Littlewood\(^1\) and Knopp\(^2,3\) on Cesaro summability of positive integral order are really theorems on equivalence of different modes of summation, veiled because they are put in an asymmetrical form. The object of this paper is to prove a general equivalence theorem in a general field of summability in a symmetrical form, from which the theorems cited above, and a number of other known theorems, such as the equivalence of Cesaro and Hölder summability of the same positive integral order, Okada's and Knopp's\(^4\) and Narumi's\(^5\) extensions of Mercer's limit theorem, etc., can all be deduced either as particular cases or as immediate consequences. This theorem appears to us to show up the structure of equivalence in fields of summability such as the ones dealt with here (a generalisation of the Cesaro-Hölder field) in a much clearer way than the theorems cited above.

After proving the general theorem we indicate briefly how some of the more important of the theorems cited above can be deduced from this theorem and, in some cases, extended. Also the generalisation of Hardy and Littlewood's Tauberian theorem in the Cesaro-Hölder field, appropriate to the general field of summability dealt with here, is given as Theorem III.

§ 2. Notations

Let \(\phi_n > 0\), \(\Sigma (1/\phi_n)\) divergent, and \(\phi_n \to \infty\) as \(n \to \infty\).

\[\delta a_n = a_n - a_{n-1}, \quad \delta^2 a_n = \delta (\delta a_n), \text{ etc.};\]

\[R(x) = \text{real part of } x.\]

\[L_n(p) = n \left( 1 + \frac{p}{\phi_r} \right)\]

the initial value of \(r\) in the product being taken so large that none of the factors is zero or negative.

\(^1\) J. Mys. Univ., Vol. III] 57

\(^2\) BI
The symbols \( s_n/(\phi_n \delta + k) \), \( s_n/(\phi_n \delta + k_1)(\phi_n \delta + k_2) \), etc., stand for the general solution of the difference equations
\[
(\phi_n \delta + k) y_n = s_n,
\]
\[
(\phi_n \delta + k_1)(\phi_n \delta + k_2) y_n = s_n, \text{ etc.}
\]

We say that the sequence \( s_n \) is summable \((\phi, k)\), if one solution of the difference equation of order \( k \),
\[
(\phi_n \delta + a_1)(\phi_n \delta + a_2) \cdots (\phi_n \delta + a_k) y_n = a_1 a_2 \cdots a_k \cdot s_n
\]
(2.1)
tends to \( s \) as \( n \to \infty \), \( a_1, a_2, \cdots, a_k \) being arbitrary non-zero constants, and indicate this by the symbol*
\[
s_n \to s, (\phi, k).
\]
(2.2)

§ 3. Statement of the Theorems

THEOREM I. If one solution of
\[
(\phi_n \delta + a_1)(\phi_n \delta + a_2) \cdots (\phi_n \delta + a_k) y_n = s_n
\]
(3.1)
is of order \( O \lfloor L_n (p) \rfloor \) or \( O \lfloor L_n (p) \rfloor \), then one solution of
\[
(\phi_n \delta + \beta_1)(\phi_n \delta + \beta_2) \cdots (\phi_n \delta + \beta_k) y_n = s_n
\]
(3.2)
will also be of order \( O \lfloor L_n (p) \rfloor \) or \( o \lfloor L_n (p) \rfloor \) respectively, provided that
(i) \( k' \geq k \), (ii) \( R (\beta_r) \neq 0, r = 1 \cdots k' \), and
(iii) \( R (p + \beta_r) \neq 0, r = 1 \cdots k' \).
(3.3)

Further if, in addition to (3.3),
\[
R (\beta_r) > 0, \ r = 1 \cdots k',
\]
(3.4)
then all solutions of (3.2) will necessarily be \( O \lfloor L_n (p) \rfloor \) or \( o \lfloor L_n (p) \rfloor \). An important special case arises when \( p = 0 \) and one solution of (3.1) tends to \( \lambda \) as \( n \to \infty \), and we then have the following theorem:—

THEOREM I\(_1\). If one solution of (3.1) tends to \( \lambda \) as \( n \to \infty \), then one or all solutions of (3.2) tend to \( \frac{\pi a_r}{\pi \beta_r} \lambda \), according as the condition (3.3) only or (3.3) and (3.4) are satisfied.†

The following theorems are natural generalisations of the corresponding theorems in the Cesaro field:—

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* The justification for this notation will be obvious from the main theorems.
† In Theorem I, the hypothesis \( \phi_n \to \infty \) will not be necessary if all the \( \beta \)'s are real and positive.
THEOREM II. If \( s_n \to s \) \((\phi, k)\), then \( \phi_n \Delta s_n \to 0 \) \((\phi, k + 1)\).

THEOREM III. If \( s_n \to s \) \((\phi, k)\) and \( \phi_n \Delta s_n \geq -\alpha \) \((\alpha > 0)\), then
\( s_n \to s \) in the ordinary sense \([i.e., (\phi, 0)]\).

Theorem II is an immediate consequence of Theorem I₁ and Lemma IV below.

Theorem III can be proved by using the known theorem for \( k = 1 \) and Theorem I₁.

§ 4. Preliminary Lemmas

Lemma I. If \( f_n (k_1, k_2) = L_n (k_1) \cdot L_n (k_2) \), then, as \( n \to \infty \)
\[
\begin{align*}
\text{if } R (k_1 + k_2) > 0, & \quad |f_n| \to \infty; \quad & \\
\text{if } R (k_1 + k_2) < 0, & \quad |f_n| \to 0; \quad & \\
\end{align*}
\]
(4.1)
in both cases \( |f_n| \) being monotonic for sufficiently large \( n \), and
\[
\frac{|\Delta f_n|}{\Delta |f_n|} \to \frac{|k_1 + k_2|}{R (k_1 + k_2)}.
\]
(4.2)

Lemma II. If \( f_1 (n) = \sum \frac{\delta L_r (k)}{L_r (k)} \)
\( f_2 (n) = \sum \frac{\delta L_r}{L_r} \),

\[
f_\phi (n) = \sum \frac{\delta L_r}{L_r} f_{\phi - 1} (r),
\]
then
\[
f_\phi (n) = O (\log |L_n|)^\phi \text{, and } \log |L_n| \sim k \sum (1/\phi_n).
\]
(4.3)
The initial value of \( r \) in each of the sums is to be chosen so large that none of the \( L_r \)'s is zero. The proof of both these lemmas is elementary.

Lemma III. The general value of \( s_n / (\phi_n \delta + k) \) is given by
\[
\frac{1}{k^\phi L_n} \sum_{n_1 = n}^{n = n_1} \frac{\delta L_{n_1}}{L_{n_1}} \frac{\delta L_{n_2}}{L_{n_2}} \ldots \frac{\delta L_{n_\phi}}{L_{n_\phi}} = \sum_{n_1 = n + 1}^{\infty} \frac{\delta L_{n_1}}{L_{n_1}} \frac{\delta L_{n_2}}{L_{n_2}} \ldots \frac{\delta L_{n_\phi}}{L_{n_\phi}} \delta L_{n_\phi} s_{n_\phi}
\]
\[
+ \frac{1}{L_n} \{ A_0 + A_1 f_1 (n) + \ldots + A_{\phi - 1} f_{\phi - 1} (n) \}
\]
where \( L_n = L_n (k) \) and \( A_0, \ldots, A_{\phi - 1} \) are arbitrary constants.
(4.4)

It is also given by
\[
\frac{1}{k^\phi L_n} \sum_{n_1 = n + 1}^{\infty} \frac{\delta L_{n_1}}{L_{n_1}} \frac{\delta L_{n_2}}{L_{n_2}} \ldots \frac{\delta L_{n_\phi}}{L_{n_\phi}} = \sum_{n_1 = n + 1}^{\infty} \frac{\delta L_{n_1}}{L_{n_1}} \frac{\delta L_{n_2}}{L_{n_2}} \ldots \frac{\delta L_{n_\phi}}{L_{n_\phi}} \delta L_{n_\phi} s_{n_\phi}
\]
\[
+ \frac{1}{L_n} \{ A_0 + A_1 f_1 (n) + \ldots + A_{\phi - 1} f_{\phi - 1} (n) \}
\]
(4.5)
provided that the first part of (4.5) is convergent.
(4.4) and (4.5) are established by repeated application of easily provable result that

\[
\frac{\partial}{\partial x} \left( \phi_n \delta + k \right) = \frac{1}{k L_n} \sum_{n=1}^{\infty} \delta L_n \mu_n + \frac{A_0}{L_n} = \frac{1}{k L_n} \sum_{n=1}^{\infty} \delta L_n \mu_n + \frac{A_0}{L_n}
\]

provided that in the latter expression the first part is convergent.

**Lemma IV.** The operators \((\phi_n \delta + k_1), (\phi_n \delta + k_2)\) and \(1/(\phi_n \delta + k_3)\) are commutative with each other \((k_1, k_2, k_3\) being constants). If \(f(x) = P_1(x)/Q_1(x)\) where \(P_1, Q_1\) are polynomials in \(x\), and \(P_1/Q_1 = P(x) + \sum_{q=1}^{N} \sum_{p=1}^{M} A_{pq}(x + \frac{k}{p})\) where \(P\) is a polynomial in \(x\) and \(A_{pq}, k_p\) are all constants, then

\[
f(\phi_n \delta) = P(\phi_n \delta) + \sum_{q=1}^{N} \sum_{p=1}^{M} \frac{A_{pq}}{(\phi_n \delta + k_p)^q}.
\]

The first part of the lemma is proved by making use of (4.6), and the second part follows from the first as in the case of the differential operator \(x \frac{d}{dx}\).

**Lemma V.** If \(R(k) \neq 0\) and \(s_n\) is \(O[L_n(p)]\) or \(o[L_n(p)]\) and \(R(p + k) \neq 0\) then one solution of \(s_n / (\phi_n \delta + k)\) will be \(O[L_n(p)]\) or \(o[L_n(p)]\) respectively and if \(R(k) > 0\), all solutions of \(s_n / (\phi_n \delta + k)\) will necessarily be \(O[L_n(p)]\) or \(o[L_n(p)]\) respectively.

**Proof.**—Suppose \(R(k) > 0\). Let \(s_n = \mu_n L_n(p)\). By (4.6)

\[
\frac{\partial}{\partial x} \left( \phi_n \delta + k \right) = \frac{1}{k L_n(k)} \sum_{n=1}^{\infty} \delta L_n (k) \mu_n L_n(p) + A_0 L_n(k)
\]

or

\[
\frac{1}{k L_n(k)} \sum_{n=1}^{\infty} \delta L_n (k) \mu_n L_n(p) + A_0 L_n(k).
\]

By Lemma I (when \(k_1\) or \(k_2\) = 0), \(|L_n(k)| \rightarrow \infty\), so that we need consider only the first part of (4.8) or (4.9). In case \(R(k + p) > 0\), consider (4.8).

Now

\[
|\delta L_n (k) \mu_n L_n(p)| \leq \frac{k}{\phi_n} L_{n-1}(k) \mu_n L_n(p)
\]

so that \(\delta L_n (k) \mu_n L_n(p)\) will be \(O[L_n(p) L_n(k)]\) or \(o[L_n(p) L_n(k)]\) according as \(\mu_n\) is \(O(1)\) or \(o(1)\), and hence

\[
s_n / (\phi_n \delta + k) = O[L_n(p)] \text{ or } o[L_n(p)].
\]
If \( R (k + p) < 0 \) we obtain the same result by considering (4.9) and in case \( R (k) < 0 \) we establish the lemma by taking \( A_0 = 0 \).

**Lemma VI.** If \( R (k) \neq 0 \) and \( s_n \) is \( O \left[ L_n (p) \right] \) or \( o \left[ L_n (p) \right] \), then so is one solution of \( s_n / (\phi_n \delta + k)^q \), where \( q \) is any positive integer; and all solutions will be so if \( R (k) > 0 \).

This is proved by repeated application of Lemma V.

**Lemma VII.** If \( R (k) \neq 0 \) and \( s_n \to \lambda \) as \( n \to \infty \), then one solution of \( s_n / (\phi_n \delta + k)^q \) tends to \( \lambda / k^q \) when \( n \to \infty \); and all solutions will be so if \( R (k) > 0 \).

**Proof.**—Let \( (\phi_n \delta + k)^q \) \( y_n = s_n \). Putting \( y_n = z_n + \frac{\lambda}{k^q} \) we have
\[
(\phi_n \delta + k)^q \ z_n = s_n - \lambda = o \quad (1).
\]
Then by Lemma VI, one solution or all solutions \( z_n \) will be \( o \quad (1) \) according as \( R (k) \leq 0 \) so that
\[
y_n \to \frac{\lambda}{k^q} \text{ as } n \to \infty.
\]

§ 5. Proof of Theorems I and I₁

Let one solution of (3.1) be \( y_{1n} \) of order \( O \left[ L_n (p) \right] \) or \( o \left[ L_n (p) \right] \).

Then (3.2) can be written as
\[
\Pi \left( \phi_n \delta + \beta_r \right) y_n = \Pi \left( \phi_n \delta + \alpha_r \right) y_{1n},
\]
or
\[
y_n = \Pi \left( \phi_n \delta + \beta_r \right) y_{1n} \quad (5.1)
\]
where in (5.1) \( \beta_1 \cdots \beta_f \) are the different \( \beta \)'s occurring \( k_1 \cdots k_f \) times such that \( k_r \), \( k' \) are the different \( \beta \)'s occurring \( k_1 \cdots k_f \) times such that \( \sum \beta_r = k' \) and \( A_{r \rho} \) are constants. By applying Lemma VI to (5.1) or (5.2) we establish Theorem I. By applying Lemma VII to (5.1) or (5.2) and making use of the fact that
\[
\sum \frac{A_{r \rho}}{\beta_r \rho} \text{ or } 1 + \sum \frac{A_{r \rho}'}{\beta_r \rho} = \frac{\pi \alpha_r}{\pi \beta_r} \quad (5.3)
\]
we prove Theorem I₁.
§ 6. Applications

(i) Equivalence of Cesaro-Hölder summability of the same positive integral order, and its extension. If \( C_n^k \) and \( H_n^k \) are the Cesaro and Hölder \( k^{th} \) means of the sequence \( n \), it is easy to prove that

\[
\begin{align*}
\pi (n\delta + r) C_n^k &= k! s_n, \\
(n\delta + 1)^k H_n^k &= s_n.
\end{align*}
\]

By writing \( C_n^k = C_n^k / k! \), (6.1) becomes

\[
\pi (n\delta + r) C_n^k = s_n. \tag{6.3}
\]

We see at once from (6.2) and (6.3) that the equivalence of C, H summability of order \( k \) is a particular case of Theorem I, with \( \phi_n = n \).

From Theorem I, we can derive the following generalisation of the equivalence:

If \( \alpha = -1, -2, \cdots \), and \( C_n^k \) (or \( H_n^k \)) is of order \( O(n^\alpha) \) or \( o(n^\alpha) \), then so is \( H_n^k \) (or \( C_n^k \)).

(ii) On the theorem of Knopp.\(^3\) We shall here shew that this is a particular case of Theorem I, and by using Theorem I we shall shew how it can be generalised.

Let

\[
y_n = \left\{ \sum_{r=0}^{n} \left( \frac{n+k-r}{k} \right) \left( \frac{p+r}{v} \right) s_v \right\} / \left( (n+k+p+1) \right). \tag{6.4}
\]

Then the theorem is as follows: "If for a particular positive integral value \( k_1 \) of \( k \) and a positive integral value \( p_1 \) of \( p \), \( y_n \to s \), then the same will be true for any \( k \geq k_1 \) and \( p \) any integer \( \geq 0 \)."

Now (6.4) implies

\[
y_n = \sum_{r=1}^{k+1} \left( \frac{(n+k+r)}{\Gamma(p+r+1)} \Gamma(p+1) \right) s_n. \tag{6.5}
\]

From (6.5) we see at once that Knopp's theorem is a particular case of Theorem I. In this form it is obvious that the restriction on \( p \) in Knopp's theorem can be replaced by the broader one \( p > -1 \); and expressing \( y_n \) in the form

\[
y_n = \left\{ \sum \left( \frac{n+k-v}{v} \right) \frac{\Gamma(p+v+1)}{\Gamma(v+1) \Gamma(p+1)} s_v \right\} / \left( \frac{\Gamma(n+k+p+2)}{\Gamma(n+1) \Gamma(k+p+1)} \right) \tag{6.6}
\]

corresponding to Theorem I, we have the following extension of this theorem: If for a particular value \( p_1 \) of \( p \), and a positive integral value \( k_1 \) of \( k \),

\[
y_n = O(n^\alpha) \text{ or } o(n^\alpha)
\]
then this will still be true for any \( p > -1 \) and any integer \( k \geq k_1 \), provided that \( R(\alpha + p + 1), R(\alpha + p + 2) \cdots R(\alpha + p + k + 1) \) are all \( \neq 0 \).

(iii) Hardy-Littlewood’s and Knopp’s conditions for \((C, r)\) summability. In terms of the difference-operator \( n\delta \), the theorem A of Hardy-Littlewood,\(^1\) and Satz. 4 of Knopp\(^2\) can be expressed as follows:

**Hardy-Littlewood’s Theorem.** If \( s_n \to s (C, r) \), then one solution of

\[
\frac{s_n}{(n\delta - 1)(n\delta + 1)(n\delta + 2) \cdots (n\delta + r - 1)}
\]

(6·7)

tends to \( s/(r - 1) \) and conversely:

**Knopp’s Theorem.** If \( s_n \to s (C, r) \), then one solution of

\[
\frac{ks_n}{(n\delta - k)(n\delta + 1) \cdots (n\delta + r - 1)}
\]

(6·8)

tends to \( s/(r - 1) \) and conversely:

It is obvious that both theorems are particular cases of Theorem I.

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