Note on
Path Signed Graphs

P. Siva Kota Reddy¹ and M. S. Subramanya²

Department of Studies in Mathematics
University of Mysore, Manasagangotri
Mysore 570 006, India

E-mail:¹ reddy_math@yahoo.com; ² subramanya_ms@rediffmail.com

Abstract

Data in the social sciences can often modeled using signed graphs, graphs where every edge has a sign + or −, or marked graphs, graphs where every vertex has a sign + or −. The path graph \( P_k(G) \) of a graph \( G \) is obtained by representing the paths \( P_k \) in \( G \) by vertices whenever the corresponding paths \( P_k \) in \( G \) from a path \( P_{k+1} \) or a cycle \( C_k \). In this note, we introduce a natural extension of the notion of path graphs to the realm of signed graphs. It is shown that for any signed graph \( S \), \( P_k(S) \) is balanced. The concept of a line signed graph is generalized to that of a path signed graphs. Further, in this note we discuss the structural characterization of path signed graphs. Also, we characterize signed graphs which are switching equivalent to their path signed graphs \( P_3(S) \) (\( P_4(S) \)).

2000 Mathematics Subject Classification : 05C 22

KEYWORDS AND PHRASES : Signed graphs, Balance, Switching, Line signed graphs, Path signed graphs, Negation.

1 Introduction

For standard terminology and notion in graph theory we refer the reader to West [11]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.
A signed graph is an ordered pair $S = (G, \sigma)$, where $G = (V, E)$ is a graph called underlying graph of $S$ and $\sigma : E \to \{+, -\}$ is a function. A signed graph $S = (G, \sigma)$ is balanced if every cycle in $S$ has an even number of negative edges (See [4]). Equivalently, a signed graph is balanced if product of signs of the edges on every cycle of $S$ is positive.

A marking of $S$ is a function $\mu : V(G) \to \{+, -\}$; A signed graph $S$ together with a marking $\mu$ is denoted by $S_\mu$.

The following characterization of balanced signed graphs is well known.

**Proposition 1.** (E. Sampathkumar [8]) A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exist a marking $\mu$ of its vertices such that each edge $uv$ in $S$ satisfies $\sigma(uv) = \mu(u)\mu(v)$.

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking $\mu$ of a signed graph $S$. Switching $S$ with respect to a marking $\mu$ is the operation of changing the sign of every edge of $S$ to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $S_\mu(S)$ and is called $\mu$-switched signed graph or just switched signed graph. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be isomorphic, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if $uv$ is an edge in $G$ then $f(u)f(v)$ is an edge in $G'$) such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further, a signed graph $S_1 = (G, \sigma)$ switches to a signed graph $S_2 = (G', \sigma')$ (or that $S_1$ and $S_2$ are switching equivalent) written $S_1 \sim S_2$, whenever there exists a marking $\mu$ of $S_1$ such that $S_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be cycle isomorphic (see [12]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle $Z$ in $S_1$ equals to the sign of $\phi(Z)$ in $S_2$. The following result will be useful in our further investigation (See [12]):

**Proposition 2.** (T. Zaslavsky [12]) Two signed graphs $S_1$ and $S_2$ with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.

## 2 Path Signed Graphs

Broersma and Hoede [2] generalized the concept of line graphs to that of path graphs. Let $P_k$ and $C_k$ denote a path and a cycle with $k$ vertices, respectively.
Denote $\Pi_k(G)$ the set of all paths of $G$ on $k$ vertices ($k \geq 1$). The path graph $P_k(G)$ of a graph $G$ has vertex set $\Pi_k(G)$ and edges joining pairs of vertices that represent two paths $P_k$, the union of which forms either a path $P_{k+1}$ or a cycle $C_k$ in $G$. A graph is called a $P_k$-graph, if it is isomorphic to $P_k(H)$ for some graph $H$. If $k = 2$, then the $P_2$-graph is exactly the line graph. The way of describing a line graph stresses the adjacency concept, whereas the way of describing a path graph stresses concept of the path generation by consecutive paths.

For $P_3$-graphs, Broersma and Hoede [2] gave a solution to the characterization problem, which contained flaw. Later, Li and Lin [6] presented corrected form of the characterization of $P_3$-graphs. For $k \geq 4$, the problems becomes more difficult. Although the determination and characterization problems for $P_k$-graphs for $k \geq 4$ have not been completely solved.

We extend the notion of $P_k(G)$ to the realm of signed graphs. In a signed graph $S = (G, \sigma)$, for any $A \subseteq E(G)$ the sign $\sigma(A)$ is the product of the signs on the edges of $A$. The path signed graph $P_k(S) = (P_k(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $P_k(G)$ called path graph and sign of any edge $e = P_kP_k'$ in $P_k(S)$ is $\sigma'(P_kP_k') = \sigma(P_k)\sigma(P_k')$. Further, a signed graph $S = (G, \sigma)$ is called path signed graph, if $S \cong P_k(S')$, for some signed graph $S'$. We now gives a straightforward, yet interesting, property of path signed graphs.

**Proposition 3.** For any signed graph $S = (G, \sigma)$, its path signed graph $P_k(S)$ is balanced.

**Proof.** Since sign of any edge $\sigma'(e = P_kP_k')$ in $P_k(S)$ is $\sigma(P_k)\sigma(P_k')$, where $\sigma$ is the marking of $P_k(S)$, by Proposition 1, $P_k(S)$ is balanced. \qed

**Remark:** For any two signed graphs $S$ and $S'$ with same underlying graph, their path signed graphs are switching equivalent.

In [3], the author defined line signed graph of a signed graph $S = (G, \sigma)$ as follows: The line signed graph of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge $ee'$ in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$ (see also, E. Sampathkumar et al. [9]). Hence, we shall call a given signed graph $S$ a line signed graph if it is isomorphic to the line signed graph $L(S')$ of some signed graph $S'$. By the definition of path signed graphs, we observe that $P_2(S) = L(S)$.

**Corollary 4.** For any signed graph $S = (G, \sigma)$, its $P_2(S)$ (= $L(S)$) is balanced.

In [10], the authors obtain structural characterization of line signed graphs as follows:
Proposition 5. (E. Sampathkumar et al. [10])
A signed graph $S = (G, \sigma)$ is a line signed graph (or $P_2$-signed graph) if, and only if, $S$ is balanced and $G$ is a line graph (or $P_2$-graph).

Proof. Suppose that $S$ is balanced and $G$ is a line graph. Then there exists a graph $H$ such that $L(H) \cong G$. Since $S$ is balanced, by Proposition 1, there exists a marking $\mu$ of $G$ such that each edge $uv$ in $S$ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge $e$ in $H$, $\sigma'(e)$ is the marking of the corresponding vertex in $G$. Then clearly, $L(S') \cong S$. Hence $S$ is a line signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a line signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $L(S') \cong S$. Hence $G$ is the line graph of $H$ and by Corollary 4, $S$ is balanced. 

We strongly believe that the above Proposition can be generalized to path signed graphs $P_k(S)$ for $k \geq 3$. Hence, we pose it as a problem:

Problem 6. If $S = (G, \sigma)$ is a balanced signed graph and its underlying graph $G$ is a path graph, then $S$ is a path signed graph.

We now characterize those signed graphs that are switching equivalent to their $P_3$ ($P_4$)-signed graphs. In the case of graphs the following results is due to Broersma and Hoede [2] and Li and Zhao [7] respectively.

Proposition 7. (Broersma and Hoede [2])
A connected graph $G$ is isomorphic to its path graph $P_3(G)$ if, and only if, $G$ is a cycle.

Proposition 8. (Li and Zhao [7])
A connected graph $G$ is isomorphic to its path graph $P_4(G)$ if, and only if, $G$ is a cycle of length at least 4.

Proposition 9. For any connected signed graph $S = (G, \sigma)$ satisfies

(i) $S \sim P_3(S)$ if, and only if, $S$ is a balanced signed graph on a cycle.
(ii) $S \sim P_4(S)$ if, and only if, $S$ is a balanced signed graph on a cycle of length at least 4.

Proof. (i) Suppose that $S \sim P_3(S)$. This implies, $G \cong P_3(G)$ and hence by Proposition 7, we see that the graph $G$ must be a cycle. Now, if $S$ is any signed graph on a cycle, Proposition 3 implies that $P_3(S)$ is balanced and hence if $S$ is unbalanced, $P_3(S)$ being balanced cannot be switching equivalent to $S$ in accordance with Proposition 2. Therefore, $S$ must be balanced.

Conversely, suppose that $S$ is a balanced signed graph on a cycle. Then, since $P_3(S)$ is balanced as per Proposition 3 and since $P_3(G) \cong G$, the result follows from Proposition 2.

Similarly, we can prove (ii) using Proposition 8. 

\[\square\]
The notion of negation $\eta(S)$ of a given signed graph $S$ defined in [5] as follows: $\eta(S)$ has the same underlying graph as that of $S$ with the sign of each edge opposite to that given to it in $S$. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in $S$ while applying the unary operator $\eta(.)$ of taking the negation of $S$.

For a signed graph $S = (G, \sigma)$, the $P_k(S)$ is balanced (Proposition 3). We now examine, the condition under which negation of $P_k(S)$ (i.e., $\eta(P_k(S))$) is balanced.

**Proposition 10.** Let $S = (G, \sigma)$ be a signed graph. If $P_k(G)$ is bipartite then $\eta(P_k(S))$ is balanced.

**Proof.** Since, by Proposition 3, $P_k(S)$ is balanced, then every cycle in $P_k(S)$ contains even number of negative edges. Also, since $P_k(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle $C$ in $P_k(S)$ are also even. This implies that the same thing is true in negation of $P_k(S)$. Hence $\eta(P_k(S))$ is balanced. □

Proposition 8 provides easy solutions to three other signed graph switching equivalence relations, which are given in the following results.

**Corollary 11.** For any signed graph $S = (G, \sigma)$,

(i) $\eta(S) \sim P_3(S)$ if, and only if, $S$ is an unbalanced signed graph on any odd cycle.

(ii) $\eta(S) \sim P_4(S)$ if, and only if, $S$ is an unbalanced signed graph on any odd cycle of length at least 5.

**Corollary 12.** For any signed graph $S = (G, \sigma)$ and for any integer $k \geq 1$, $P_k(\eta(S)) \sim P_k(S)$.

**Acknowledgement**

The authors thankful to Department of Science and Technology, Government of India, New Delhi for the financial support under the project grant SR/S4/MS:275/05.

**References**


