TOTAL NUMBER OF SPECIAL KINDS OF NEIGHBOURHOOD SETS OF GRAPHS

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ABSTRACT

This paper is concerned with special kinds of neighbourhood sets of graphs \( P_n \) and \( C_n \). The neighbourhood set, the split neighbourhood sets of a graph are considered using a recurrence relation, we give the number of all neighbourhood sets mentioned for the graphs \( P_n \) and \( C_n \).

INTRODUCTION:

In this paper, we consider the path \( P_n \) and cycle \( C_n \) graphs of \( n \) vertices as graphs with \( V(P_n) = \{x_1, x_2, \ldots, x_n\} \), \( E(P_n) = \{x_i, x_{i+1} : 1 \leq i \leq n-1\} \) for \( n \geq 3 \) and \( V(C_n) = \{x_1, x_2, \ldots, x_n\} \), \( E(C_n) = \{x_i, x_{i+1} : 1 \leq i \leq n-1, x_n, x_1\} \) for \( n \geq 3 \). For \( v \in V \), the closed neighbourhood of \( v \) is \( N[v] = \{u \in v : uv \in E(G)\} \cup \{v\} \).

A subset \( S \) of \( V(G) \) is a neighbourhood set of \( G \) if \( G = \bigcup_{v \in S} \langle N(v) \rangle \) where \( \langle N(v) \rangle \) is the subgraph of \( G \) induced by \( N[v] \). The neighbourhood number \( \eta(G) \) of \( G \) is the minimum cardinality of a neighbourhood set of \( G \).

Neighbourhood sets of \( P_n \) and \( C_n \):

The results considered in this section will be used to determine the total number of split neighbourhood sets of \( P_n \) and \( C_n \). To determine the number of all neighbourhood sets of a path of \( n \) vertices, for all \( n \geq 1 \), we introduce the following notations.

\[ N(P_n) = \{D \subseteq V(G) : D \text{ is a neighbourhood set of } P_n\} \]

\[ N_1(P_n) = \{D \in N(P_n) : X_n \in D\} \]

\[ N_2(P_n) = N(P_n) - N_1(P_n) \]

Denoting the cardinalities of the families \( N(P_n), N_1(P_n) \) and \( N_2(P_n) \) respectively by \( N(P_n) \), \( N_1(P_n) \), \( N_2(P_n) \). We obtain the following equality,

\[ N(P_n) = N_1(P_n) + N_2(P_n), \quad n \geq 1 \quad \text{(1)} \]

of course \( N(P_n) \) is the total number of neighbourhood sets of a path \( P_n \) on \( n \) vertices. It is easy to see that \( N(P_1) = 1 \), \( N(P_2) = 3 \), \( N(P_3) = 5 \). These inequalities are the initial conditions for the recurrence relation given in the following theorem.
Theorem 1: For \( n \geq 3 \),
\[
\mathcal{N}(P_{n+1}) = \mathcal{N}(P_n) + \mathcal{N}(P_{n-1})
\]

Proof: Assume the \( \mathcal{N} \) is a neighborhood set of \( P_{n+1} \) for \( n \geq 3 \). Then we have three possibilities:

1. If \( X_n, X_{n+1} \in D \), then \( D - \{X_{n+1}\} \in \mathcal{N}_1(P_n) \).
2. If \( X_n \in D \) and \( X_{n+1} \notin D \), then \( D \in \mathcal{N}_1(P_n) \).
3. If \( X_n \notin D \) and \( X_{n-1}, X_{n+1} \in D \), then \( D - \{X_{n+1}\} \in \mathcal{N}_2(P_n) \).

Thus, the number \( \mathcal{N}(P_{n+1}) \) of all neighborhood sets of \( P_{n+1} \) is described by the following equality.

\[
\mathcal{N}(P_{n+1}) = \mathcal{N}_1(P_n) + \mathcal{N}_1(P_{n-1}) + \mathcal{N}_2(P_{n-1})
\]

Applying these equalities in equation (3) to the first of the sum in equation (2), we obtain

\[
\mathcal{N}(P_{n+1}) = \mathcal{N}_1(P_{n-1}) + \mathcal{N}_2(P_{n-1}) + \mathcal{N}_1(P_n) + \mathcal{N}_2(P_n)
\]

Hence \( \mathcal{N}(P_{n+1}) = \mathcal{N}(P_{n-1}) + \mathcal{N}(P_n) \).

The following results concern the total number of neighborhood sets of the cycle \( C_n \) on \( n \) vertices for \( n \geq 4 \). First we introduce the following notations:

\( \mathcal{N}(C_n) = \{D \subseteq V(G) : \mathcal{N} \text{ is a neighborhood set of } C_n \} \).
\( \mathcal{N}_0(c_n) = \{D \in \mathcal{N}(C_n) : (X_1 \in D \text{ and } X_n \notin D) \text{ or } (X_1 \notin D \text{ and } X_n \in D) \} \)
\( \mathcal{N}_1(c_n) = \{D \in \mathcal{N}(C_n) : X_1, X_n \in D \} \)

By \( \mathcal{N}(C_n) \), \( \mathcal{N}_0(c_n) \) and \( \mathcal{N}_1(c_n) \) we mean the cardinalities of families \( \mathcal{N}(C_n) \), \( \mathcal{N}_0(c_n) \) and \( \mathcal{N}_1(c_n) \) respectively. Using these numbers, we obtain the following equality.

\[
\mathcal{N}(C_n) = \mathcal{N}_0(c_n) + \mathcal{N}_1(c_n), \quad n \geq 4
\]

It is easy to check that \( \mathcal{N}(C_4) = 7 \) and \( \mathcal{N}(C_5) = 11 \).

Theorem 2: For \( n \geq 5 \),
\[
\mathcal{N}(C_{n+1}) = \mathcal{N}(C_{n-1}) + \mathcal{N}(C_n)
\]

Proof: Let \( \mathcal{N} \) be a neighborhood set of \( C_n \) for \( n \geq 5 \). Consider the following cases:

1. If \( X_1 \notin D \) and \( X_2, X_n \in D \), then \( D \in \mathcal{N}_1(H_1) \) where
   \( V(H_1) = V(C_n) - \{X_1\} \) and \( E(H_1) = \{E(C_n) - \{X_nX_1, X_1X_2\}\} \cup \{X_nX_2\} \)
Hence \( H_1 = C_{n-1} \)

(ii) \( \text{If} X_n \notin D \quad \text{and} \quad X_1, X_{n-1} \in D \quad \text{then} \quad D \in \mathbb{N}_{01}(H_2) \), where
\[
V(H_2) = V(C_n) - (X_n) \quad \text{and} \quad E(H_2) = \{ E(C_n) - (X_{n-1}X_n, X_nX_1) \} \cup (X_{n-1}X_1)
\]

Hence \( H_2 = C_{n-1} \)

(iii) \( \text{If} X_1, X_n \in D \quad \text{and} \quad X_2 \notin D \quad \text{then} \quad D \in \mathbb{N}_{01}(H_1) \)

(iv) \( \text{If} X_1, X_2, X_n \in D \quad \text{then} \quad D \in \mathbb{N}_{11}(H_1) \)

Hence from case (i),(ii),(iii) and (iv) we have that
\[
\mathbb{N}_{01}(C_n) = \mathbb{N}_{11}(H_1) + \mathbb{N}_{11}(H_2)
\]
\[
\mathbb{N}_{01}(C_n) = 2\mathbb{N}_{11}(C_{n-1})
\]
\[
\mathbb{N}_{11}(C_n) = \mathbb{N}_{01}(C_{n-1})/2 + \mathbb{N}_{11}(C_{n-1})
\]
\[
\mathbb{N}(C_n) = \mathbb{N}_{01}(C_{n}) + \mathbb{N}_{11}(C_{n})
\]
\[
\mathbb{N}(C_n) = 2\mathbb{N}_{11}(C_{n-1}) + \mathbb{N}_{01}(C_{n-1})/2 + \mathbb{N}_{11}(C_{n-1})
\]
\[
\mathbb{N}(C_n) = (2\mathbb{N}_{01}(C_{n-2})/2 + \mathbb{N}_{11}(C_{n-2})) + \mathbb{N}_{01}(C_{n-1})/2 + \mathbb{N}_{11}(C_{n-1})
\]
\[
\mathbb{N}(C_n) = \mathbb{N}_{01}(C_{n-2}) + 2\mathbb{N}_{11}(C_{n-2}) + 1/2(2\mathbb{N}_{11}(C_{n-2})) + \mathbb{N}_{11}(C_{n-1})
\]
\[
\mathbb{N}(C_n) = \mathbb{N}_{01}(C_{n-2}) + \mathbb{N}_{01}(C_{n-1}) + \mathbb{N}_{11}(C_{n-2}) + \mathbb{N}_{11}(C_{n-1})
\]

From equation (4) the above equation reduces to
\[
\mathbb{N}(C_n) = \mathbb{N}_{01}(C_{n-2}) + \mathbb{N}_{11}(C_{n-2}) + \mathbb{N}_{01}(C_{n-1}) + \mathbb{N}_{11}(C_{n-1})
\]

Split neighbourhood set of \( P_n \) and \( C_n \):

Using the numbers \( \mathbb{N}(P_n) \) and \( \mathbb{N}(C_n) \) we determine the number of split neighbourhood sets of path and the cycle of graphs on \( n \) vertices. First we give supportive theorem that characterize the split neighbourhood set of \( P_n \) and \( C_n \).

**Lemma 3:** Any neighbourhood set \( D \) of \( P_n \) \( n \geq 3 \), is a split neighbourhood set of \( P_n \) if and only if \( V(P_n) - D \neq \emptyset \), \( \langle V(P_n) - D \rangle_{(P_n)} \neq K_1 \).

**Proof:** Let \( S \) be a split neighbourhood set \( D \) of \( P_n \). According to the definition of \( S \), it follows that \( \langle V(P_n) - D \rangle \) is disconnected. Thus \( V(P_n) - D \neq \emptyset \), \( \langle V(P_n) - D \rangle_{(P_n)} \neq K_1 \), proving necessity.

For sufficiency, let \( D \) be a neighbourhood set of \( P_n \), \( n \geq 3 \), and suppose \( V(P_n) - D \neq \emptyset \), \( \langle V(P_n) - D \rangle_{(P_n)} \neq K_1 \). Since \( H = \langle V(P_n) - D \rangle_{(P_n)} \) is an induced subgraph of \( P_n \), then any connected component of \( H \) is isomorphic to the path \( P_k \), \( 1 \leq k \leq n-1 \) (where \( P_1 = K_1 \)). Let \( H_1 \) be a connected

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component of the subgraph $H$. We show that $H_1$ is not a unique connected component of $H$. First, observe that $H_1$ contains atmost two vertices. Otherwise there would exist a vertex of $H_1 \subset H = \langle V(P_n - D) \rangle_{(P_n)}$ not neighbour of $D$ in $P_n$. Consequently, $H_1 = P_1$ or $H_1 = P_2$. Hence $H$ has at least two connected components, because $H_1 \neq P_1$ or $H_1 \neq P_2$ by premise. This shows that $H = \langle V(P_n - D) \rangle_{(P_n)}$ is disconnected. Moreover, since $D$ is also a neighbourhood set of $P_n$, it is a split neighbourhood set of $P_n$, completes the proof of the theorem.

Similar to the case of $P_n$, we have a result concerning the split neighbourhood sets of $C_n$.

**Lemma 4:** Any neighbourhood set $D$ of $C_n$, $n \geq 4$ is a split neighbourhood set of $C_n$ if and only if $V(P_n) - D \neq \emptyset$, $\langle V(C_n) - D \rangle_{(C_n)} \neq K_1$.

Additionally, observe that there is only one neighbourhood set $D$ of $P_n$ such that $V(P_n) - D \neq \emptyset$, and there are exactly $n$ neighbourhood sets $D$ of $P_n$ such that $\langle V(P_n) - D \rangle_{(P_n)} \approx K_1$.

In special cases of $C_n$, we have one Neighbourhood set $D$ such that $V(C_n) - D = \emptyset$ and $n$ for $\langle V(C_n) - D \rangle_{(C_n)} \approx K_1$.

For $n \geq 3$, we introduce the notation

$S(G) = \{D \subseteq V(G) : D \text{ is a split neighbourhood set of } G\}$

$S(G) \equiv |S(G)|$

From the above, we have the following corollary, which will be used in proving theorem.

**Corollary 5:** $s(P_n) = d(P_n) - (1 + n)$ for $n \geq 3$

and $s(C_n) = d(C_n) - (1 + n)$ It is easy to see that $s(P_3) = 1$, $s(P_4) = 3$ and $s(P_5) = 7$.

**Theorem 6:** For $n \geq 5$,

$s(P_{n+1}) = s(P_{n+1}) + s(P_n) + (n-1)$

**Proof:** Let $n \geq 5$, according to corollary (3), for $P_{n+1}$ we have that

$s(P_{n+1}) = d(P_{n+1}) - (2 + n)$ since $d(P_{n+1}) = d(P_{n+1}) + d(P_n)$

by theorem (1), we obtain

$s(P_{n+1}) = d(P_{n+1}) - (2 + n)$

$= d(P_{n+1}) + d(P_n) - (2 + n)$

$= d(P_{n+1}) - n + d((P_n) - (n + 1)) - (2 + n)$

$= d(P_{n+1}) - n + d((P_n - (n + 1)) - (2 + n + n + (n + 1))$}

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since \((n+1+n-2-n=(n-1))\), it follows that
\[
d = (P_{n-1}) - n) + d((P_n) - (n+1)) + (n-1)
\]
Finally applying corollary (3) to the expressions in brackets,
\[
s(P_{n+1}) = s(P_{n-1}) + s(P_n) + (n-1)
\]

**Theorem 7:** For \(n \geq 5\),
\[
s(C_{n+1}) = s(C_{n-1}) + s(C_n) + (n-1)
\]

**Proof:** Let \(n \geq 5\), putting \(n+1\) in place of \(n\) in corollary 3, it follows
\[
s(C_{n+1}) = d(C_{n+1}) - (2+n)\) according to theorem (2).
\[
d(C_{n+1}) = d(C_{n-1}) - d(C_n)\) hence,
\[
s(C_{n+1}) = d(C_{n+1}) - (2+n)
\]
\[
= d(C_{n-1}) + d(C_n) - (2+n)
\]
\[
= d((C_{n-1}) - n + n) + d((C_n) - (n+1) + (n+1)) - (2+n)
\]
\[
= d((C_{n-1}) - n) + d((C_n) - (n+1)) + (n+1+n-2-n)
\]
since \((n+1+n-2-n=(n-1))\), it follows that
\[
d = d((C_{n-1}) - n) + d((C_n) - (n+1)) + (n-1)
\]
Finally applying corollary (3) to the expressions in brackets,
\[
s(C_{n+1}) = s(C_{n-1}) + s(C_n) + (n-1)
\]

**REFERENCES:**


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