ELLiptic function formulæ and plane cubic curves.

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1. Introduction.

The parametric representation

\[ x = \wp\left( u \right); \ y = \wp'\left( u \right) \]

is frequently employed to deduce geometrical properties of cubic curves with
the aid of elliptic function formulæ. Hadamard indicated¹ that this
procedure might be reversed and expressed the desire to see a number of
elliptic function formulæ deduced with the aid of the geometry of plane
cubic curves. Later on, a number of formulæ were deduced by Gambier² by
using involution properties of cubics and the method of residuation.

In this paper I indicate a number of other formulæ which depend on
involution properties. I also formulate Hadamard’s problem in an alterna-
tive way, viz., “Given any elliptic function formula to determine the correspond-
ing geometrical property of the cubic curve”. This formulation which is
distinct from the usual method of deducing known geometrical properties
with the aid of elliptic functions has enabled me to deduce a mixed-polar
property of the cubic which is perhaps new and can be considered
fundamental.

I indicate finally a few formulæ which depend on the method of
residuation and a metrical property related to a known elliptic function
formula.

2. Basic Theorems.

It is assumed throughout that, of the two periods, \( 2\omega \) is real and \( 2\omega' \)
purely imaginary and \( \omega, \omega', -(\omega + \omega') \) have been replaced by \( \omega_1, \omega_2, \omega_3 \),
respectively. The curve now becomes bipartite consisting of an oval and an
infinite branch having an inflexion at infinity (I) corresponding to \( u=0 \). If
P be a point \((-2u)\) on this branch, four real tangents can be drawn to the
cubic form P, the points of contact M, M₁, M₂, M₃ having parameters
\( u, u+\omega_1, u+\omega_2, u+\omega_3 \) respectively.

² Ibid., 1926, t. 54, 41 and 1930, t. 58, 220.
Among the involution properties of cubics the following two are fundamental, viz.:

(a) Corresponding points of any particular system sub tend an involution pencil at any point on the curve. This theorem is substantially equivalent to Schroeter's method\textsuperscript{3} of generation of a cubic by the points of intersection of two projective involution pencils semi-perspectively situated.

(b) Tetrads of corresponding points subtend a syzygetic pencil at any point on the curve (i.e., a pencil of tetrads containing three perfect squares or a pencil of the form \( f + \lambda H = 0 \), where \( H \) is the Hessian of the quartic).

It may also be remarked that this can be deduced from (A).\textsuperscript{4} I now consider in the following two articles certain elliptic function formulæ that can be deduced by applying (A) or (B).

3. Involution Pencil through any Point \( P \) of Curve. (See Fig. 1.)

From (A) it follows that by joining \( P \) to pairs of corresponding points on the curve, we get three types of involutions (one elliptic and two

\textsuperscript{3} Math. Ann., Bd. 5, S. 63.

Elliptic Function Formulae and Plane Cubic Curves

hyperbolic) according as the two corresponding points are both on the same branch or one on each branch. Thus PM and PM₁, PM₂, PM₃, PA₂ and PA₃; PA₁ and PI are all corresponding rays of an elliptic involution. Let \( m, m₁, m₂, m₃, m₄ \) and \( m', m'' \) denote the slopes of PM, PM₁, PM₂, PM₃ and PA₁, PA₂, PA₃ respectively. Let us associate with this pencil an involution range formed on the line \( x=1 \) by rays through O parallel to those through P. Denoting on this range by the suffix zero attached to the slopes of corresponding rays, we see that for the range considered \( m₀ \) is the centre of involution and \( m₁₀, m₂₀ \) and \( m₃₀, m₄₀ \) corresponding points.

Hence

\[
(m - m₁) (m - m₂) = (m - m₃) (m - m₄) \quad \ldots \quad \ldots \quad \ldots \quad (2)
\]

This is an elliptic function formula if we remember that

\[
m₁ = \frac{\wp'(u)}{\wp''(u)}, \text{ etc.}
\]

and \( m = \frac{\wp''(2u)}{\wp'(2u) - e₁}, \text{ etc.} \)

Other formulæ could be deduced by considering the double points of the range considered. To obtain⁵ the double lines of the elliptic involution through P, consider the point P' corresponding to P and on the same branch as P. P' will have the parameter \((-2u + \omega)\). If P'O', P'Q' and P'R, P'R' be the four tangents from P' to the curve, the lines QP'O' and RR'P will be the required double lines. The points Q and R have parameters \((u + \omega / 2)\) and \((u + \omega / 2 + \omega')\). Hence the slopes of the double lines are

\[
\frac{\wp'(2u) - \wp'(u + \omega / 2)}{\wp'(2u) - \wp(u + \omega / 2)} \quad \text{and} \quad \frac{\wp'(2u) - \wp'(u + \omega / 2 + \omega')}{\wp'(2u) - \wp(u + \omega / 2 + \omega')}
\]

say \( \mu₁ \) and \( \mu₂ \). Expressing the fact that \( m₀ \) is the mid-point of \( \mu₁₀ \mu₂₀ \) we have,

\[
\mu₁ + \mu₂ = 2m
\]

i.e.,

\[
\frac{\wp'(2u) - \wp'(u + \omega / 2)}{\wp'(2u) - \wp(u + \omega / 2)} + \frac{\wp'(2u) - \wp'(u + \omega / 2 + \omega')}{\wp'(2u) - \wp(u + \omega / 2 + \omega')} = \frac{2\wp'(2u)}{\wp'(2u) - e₁} \quad \ldots \quad (3)
\]

Again, PA₂ and PA₃ being corresponding rays of the same involution we can also write down

\[
(m - m') (m - m'') = (m - \mu₁)² = (m - \mu₂)²
\]

The first member of this expression is equal to

\[
\begin{align*}
\lbrace \wp'(2u) - \wp'(2u) \rbrace & \quad \lbrace \wp'(2u) - \wp'(2u) \rbrace \\
\lbrace \wp'(2u) - e₁ - \wp'(2u) - e₂ \rbrace & \quad \lbrace \wp'(2u) - e₁ - \wp'(2u) - e₃ \rbrace
\end{align*}
\]

= 4\lbrace \wp'(2u + \omega) - e₁ \rbrace, \text{ using the relations}

\[
\lbrace \wp'(2u + \omega) - e₁ \rbrace \lbrace \wp'(2u) - e₁ \rbrace = (e₁ - e₂)(e₁ - e₃)
\]

and \( \wp'^2(2u) = 4\lbrace \wp'(2u) - e₁ \rbrace \lbrace \wp'(2u) - e₂ \rbrace \lbrace \wp'(2u) - e₃ \rbrace \).

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⁵ See for e.g., A. Harnack, Math. Ann., Bd. 9, S. 8-10.
Hence we can write the formula
\[ \frac{\Phi'(2u)}{\Phi(2u) - \varepsilon_1} - \frac{\Phi'(u + \omega/2)}{\Phi(2u) - \Phi(u + \omega/2)} = 2 \sqrt{\Phi'(2u + \omega) - \varepsilon_1} \]  \hspace{1cm} (4)

Formula (2), (3) and (4) can also be derived by application of (B) by taking for \( f \) the binary quartic whose roots are \( m_1, m_2, m_3, m_4 \). We can deduce by this means the well-known formulæ,
\[ \frac{1}{2} \{ m_1, m_2, m_3, m_4 \} = \pm \{ \Phi(2u) - \varepsilon_1 \}^{\frac{1}{4}} \pm \{ \Phi(2u) - \varepsilon_2 \}^{\frac{1}{4}} \]
\[ \pm \{ \Phi(2u) - \varepsilon_3 \}^{\frac{1}{4}} \]  \hspace{1cm} (5)

4. Other Involution Formulæ. (See Fig. 2.)

A number of other formulæ could be deduced by applying (A) to the involution pencils through the corresponding points \( A_1 \) and \( I \). Since \( I \) is at infinity this pencil can conveniently be represented by an involution range on the X-axis, the points of the range being the feet of ordinates through corresponding points of the curve. Considering the elliptic involution through \( I \), the double points are given by the points of contact of tangents from \( A_1 \) to the curve, \textit{viz.}, the two points \( F \left\{ \Phi \left( \frac{\omega}{2} \right), 0 \right\} \) and \( F' \left\{ \Phi \left( \frac{\omega}{2} + \omega' \right), 0 \right\} \) and \( A_1 \) will be the centre of involution. Further \( A_2 \) and \( A_3 \) are also corresponding points in the same involution whose radius is therefore equal to \( \{(e_1 - e_2)(e_1 - e_2)\}^{\frac{1}{4}} \).

Since \( A_1 \) is the mid-point of \( FF' \)
\[ \Phi \left( \frac{1}{2} \omega \right) + \Phi \left( \frac{1}{2} \omega + \omega' \right) = 2e_1 \]  \hspace{1cm} (6)
Also \( FF' = \) twice the radius of the involution. Hence
\[ \Phi \left( \frac{1}{2} \omega \right) - \Phi \left( \frac{1}{2} \omega + \omega' \right) = 2 \{(e_1 - e_2)(e_1 - e_2)\}^{\frac{1}{4}} \]  \hspace{1cm} (7)

Again, if \( A_1M_1 \) meets the curve in \( M' \), this point is the reflexion of \( M \) and has parameter \( -u \). Since \( M \) and \( M_1 \) are corresponding points, \( K' \) and \( K_1 \) are corresponding points in the involution range considered.
\[ \therefore A_1K_1 \cdot A_1K' = A_1F^2 \]

but from similar triangles
\[ \frac{M_1K_1}{M'K'} = \frac{A_1K}{A_1K'} \]
\[ \therefore \frac{M_1K_1}{M'K'} = \left( \frac{A_1F}{A_1K'} \right)^2 \]
\[ i.e., \frac{\Phi'(u + \omega)}{\Phi'(u)} = \left( \frac{\Phi \left( \frac{1}{2} \omega \right)}{\Phi \left( u + \omega \right) - \Phi \left( u \right)} \right)^2 \]  \hspace{1cm} (8)

Instead of considering the elliptic involution through \( I \), let us consider the hyperbolic involution through \( A_1 \) whose corresponding rays are \( A_1M_3 \) and \( A_1M \). Also \( A_1A_3, A_1I \) and \( A_3M_2, A_3M \) are corresponding rays of this same
involution pencil. Slopes of lines $A_1A_3$ and $A_1I$ are $0$ and $\infty$. We can therefore write

\[
\frac{\Phi''(u)}{\Phi(u) - e_1} \cdot \frac{\Phi'(u + \omega_2)}{\Phi(u + \omega) - e_2} = -\frac{\Phi''(\omega/2)}{\Phi(\omega/2) - e_1} \cdot \frac{\Phi'(\omega/2 + \omega)}{\Phi(\omega + \omega) - e_1} = 4(e_3 - e_2).
\]

Equating this to the product of the slopes of $A_1M_2$ and $A_1M_3$ we deduce similarly,

\[
\frac{\Phi'(u + \omega)}{\Phi(u + \omega) - e_1} \cdot \frac{\Phi'(u + \omega + \omega')}{\Phi(u + \omega + \omega') - e_1} = 4(e_3 - e_2).
\]

By multiplication,

\[
\frac{\Pi \Phi''(u + \omega)}{\Pi \{\Phi(u + \omega) - e_1\}} = 16(e_2 - e_3)^2
\]

The denominator is equal to $(e_1 - e_2)^2(e_1 - e_3)^2$. Hence

\[
\Phi''(u) \Phi''(u + \omega_1) \Phi''(u + \omega_2) \Phi''(u + \omega_3) = 16(e_1 - e_2)^2(e_2 - e_3)^2(e_3 - e_1)^2 \quad (9)
\]


Let us start with the well-known formula of elliptic functions, viz.,

\[
\frac{\Phi''(u)}{\Phi(u)} + \frac{\Phi''(u + \omega_1)}{\Phi(u + \omega_1)} + \frac{\Phi''(u + \omega_2)}{\Phi(u + \omega_2)} + \frac{\Phi''(u + \omega_3)}{\Phi(u + \omega_3)} = 0 \quad (10)
\]

or $m_1 + m_2 + m_3 + m_4 = 0$

and try to interpret this geometrically. Let $PI_1$ and $PI_1'$ (Fig. 2) be the fourth harmonics of $I_1P'$ with respect to $PM, PM_1$ and $PM_2, PM_3$. Formula (10) is equivalent to the statement that $PI_1$ and $PI_1'$ are equally inclined to the $X$-axis or that $P(Q'P', I_1I_1')$ is a harmonic pencil. Since we can group
PM, PM₁, PM₂ and PM₃ in three different ways into pairs, this statement can be given a more general form. Consider the four tangents from P as forming a quartic; the polar line of any point on PP' with respect to this quartic is an unique line through P and might be called the polar line of PP' w.r.t. the pencil of tangents. The statement made above can be replaced by the statement that the polar line of PP' w.r.t. this pencil is the same as PQ'.

It is well known that if P and Q be two points on a cubic the polar l of P w.r.t. the polar conic of Q is the same as the polar of Q w.r.t. the polar conic of P and l is called the mixed polar of P and Q. This line passes through R, the point where PQ again meets the curve. Let us now consider the mixed polar of I and P'. The polar conic of I is made up of the inflexional tangent at infinity and the harmonic polar of I which is the X-axis. Hence the polar of P' w.r.t. this degenerate polar conic is a line parallel to the X-axis and must also pass through P. Hence the mixed polar of I and P' is the line PQ', which we have already proved to be the polar line of IP' with respect to the pencil of tangents from P to the cubic. Since only projective relations have been used in the above, we might generalise and obtain the Theorem (C): The mixed polar of two points P and Q on a cubic coincides with the polar line of PQ with respect to the pencil of four tangents to the cubic from the point R in which PQ again meets the cubic.

This geometrical theorem deduced by means of elliptical function formula (10) can also be derived directly analytically as follows:—

Let \( a_x^3 = b_x^3 = 0 \) be the equation to the cubic and \( y, \xi, z \) the points P, Q, R respectively. The equation to the four tangents drawn to the curve from \( \xi \) can be written in the form\(^6\)

\[ \phi_\xi = 4 a_x^3 b_y^2 b_z - 3 a_x^2 a_y b_x^2 b_z = 0. \]

Polarising this form twice with respect to \( y \) we can write the equation to the polar line of \( y \) with respect to \( \phi_\xi = 0 \) in the form

\[ \phi_\xi \phi_y = a_y^3 b_x b_y^2 + 3 a_y^2 a_x b_y b_x^2 - 3 a_x a_y b_y^2 a_y b_x^2 = 0 \]

or \( a_y^2 a_x b_y b_x^2 - a_x a_y b_y^2 a_y b_x^2 = 0, \) since \( a_y^3 = 0. \)

The points P, Q, R being collinear, we can write \( \xi = y + \lambda z \); with this substitution the above equation easily reduces to

\[ a_y^2 a_x b_y (b_y^2 + 2 \lambda b_y b_z + \lambda^2 b_z^2) - a_x a_y b_y^2 (a_y + \lambda a_x) (b_y + \lambda b_z) = 0. \]

i.e., \( a_y^2 a_x (b_y^2 b_z + \lambda b_y b_z^2) - \lambda a_x a_y b_z^2 b_y = 0, \) using \( b_y^3 = 0. \)

Since R lies on the curve \( b_z^3 = 0, \) or \( b_z^3 + \lambda b_y b_z^2 = 0 \) which gives

\[ b_y^2 b_z + \lambda b_y b_z^2 = 0. \]

\(^6\) See for e.g., Harnack, Math. Ann., Bd. 9, 218, where the form is subjected to a detailed investigation.
Using this equation, the polar line of \( y \) with respect to \( p_x^4 = 0 \), finally reduces to
\[
\begin{align*}
& a_x \ a_y \ a_z \cdot b_y^2 \ b_z = 0 \\
\text{or,} \\
& a_x \ a_y \ a_z = 0
\end{align*}
\]
since the form \( b_y^2 \ b_z \) cannot be equal to zero, \( R \) being distinct from \( P \).
This last is evidently the polar of \( y \) w.r.t. \( a_x^2 \ a_z = 0 \), the polar conic of \( z \) or the polar of \( z \) w.r.t. \( a_x^2 \ a_x = 0 \), the polar conic of \( y \), i.e., it is the mixed polar of \( y \) and \( z \). This proves the mixed polar theorem (C).

The mixed polar theorem can be considered fundamental and itself used to derive other elliptic function formulae. Consider, for example, the pencil \( I \) (\( M, M_1, M_2, M_3 \)). If the inflexional tangent at \( I \) cuts the harmonic polar (X-axis) at the point at infinity \( X \), then it follows from the mixed polar theorem that \( IP \) is the linear polar of \( I X \) w.r.t. this pencil. Hence it follows from the harmonic property relating to a polar line that
\[
x_m + x_{m_1} + x_{m_2} + x_{m_3} = 4x_x \\
i.e. \quad \mathcal{Q} (u) + \mathcal{Q} (u + \omega_1) + \mathcal{Q} (u + \omega_2) + \mathcal{Q} (u + \omega_3) = 4\mathcal{Q} (2u) \quad \ldots (11)
\]
another well-known elliptic function formula.


It is interesting to observe that formula (6) can be deduced by the method of residuation. If \( A_1 \) A and \( A_1 \) B be two of the tangents from \( A_1 \) such that the foci of involution \( F, F' \) are the feet of the ordinates through \( A \) and \( B \), formula (6) will have been proved if it be shown that the parabola touching the curve at \( A \) and \( B \) has its axis parallel to \( A_1 \) I, or that the group of points \( (A A A B B) \) should be residual to \( (I I) \). We have, however, \( (A A A_1) = 0 \) and \( (B B A_1) = 0 \). Hence \( (A A B B) \) is residual to \( (A_1 A_1) \) which in turn is residual to \( (I I) \) and this proves the theorem.

Consider next, the formula
\[
\frac{\mathcal{Q}''(u) - \mathcal{Q}''(u + \omega)}{\mathcal{Q}''(u + \omega)} = \frac{2}{\mathcal{Q}''(u) - e_1} \quad \ldots \quad \ldots (12)
\]
which is proved by Gambier (loc. cit., 58, p. 220) by using the method of residuation. An alternative, but equivalent, proof is obtained by observing that the points I, M, M' \( 1 \) are the points of contact of a conic with the curve since the sum of their parameters is \( \omega \). From the harmonic property associated with a conic inscribed in a triangle it readily follows that the tangents at \( M \) and \( M' \), the line MM' \( 1 \) and the inflexional tangent at infinity are harmonically situated. This shows that slope of tangent at \( M \times \) slope of tangent at \( M' = 2 \) slope of \( M M' \), which is exactly formula (12).

\footnote{My thanks are due to the Referee for drawing my attention to this, suggesting the application to formula (11) and also pointing out an alternative proof of the mixed polar theorem without the use of symbolic notation.}
Lastly, consider a group of four points on the cubic (1) the sum of whose parameters is equal to zero. This can be interpreted by saying that the parabola through these points has its axis parallel to the inflexional tangent at infinity since I corresponds to the parameter zero. We can take the equation to the parabola therefore as

\[ y = ax^2 + bx + c. \]

Considering its intersections with

\[ y^2 = 4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3) \]

we deduce the quartic whose roots are \( a, b, \gamma, \delta \) in the form

\[ (ax^2 + bx + c)^2 - 4(x - e_1)(x - e_2)(x - e_3) = a^2(x - a)(x - \beta)(x - \gamma)(x - \delta). \]

Substituting successively \( x = e_1, e_2, e_3 \) in this identity and adding the results obtained after extraction of square root and multiplication by \( (e_2 - e_3), (e_1 - e_2), (e_2 - e_1) \) respectively, we have

\[ \sum (e_2 - e_3)((a - e_1)(b - e_1)(\gamma - e_1)(\delta - e_1)) = \Pi (e_2 - e_i) \]

where

\[ a = \Phi(u_1), \text{ etc., and } u_1 + u_2 + u_3 + u_4 = 0. \]

7. A Metrical Property.

Although not connected with the central idea of this paper of associating elliptic function formulae with typical geometrical properties, I give below a theorem relating to the cubic (1) which is nevertheless interesting.

Consider formula (11) which has been derived above by means of the mixed polar theorem. This can also be deduced by ad hoc Cartesian manipulation by finding the equation giving the abscissæ of the points of intersection of the cubic and the first polar of \( P \). By adopting the same method to the case where the point \( P \) \((x', y')\) is not on the curve, I have deduced the results

\[
\begin{align*}
\sum_{r=1}^{6} \Phi(u_r) &= 6x' \\
\sum_{r=1}^{6} \Phi'(u_r) &= -6y'
\end{align*}
\]

\( u_r \) being the point of contact of a tangent from \( P \) to the curve. This leads to the

**Theorem:**—The mean centre of the points of contact of tangents drawn from any point (not on the curve) to the curve is the reflexion of the point on the X-axis.

The theorem is, however, not true when the point lies on the curve. Further, let us denote by \( S \) the expression

\[ S = 4x^3 - g_2x - g_3 - y^2. \]

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By obtaining an equation in \((x'y')\) for the points of intersection of the cubic S=0 and the first polar of any point \((x'y')\) in the plane, we can derive the result
\[
\sum_{r} \frac{\phi(u_r)}{\phi'(u_r)} = \frac{2x'y'}{4x'^3 - g_2x'^2 - g_3}.
\]

If \((x'y')\) lies on the cubic, this reduces to
\[
\sum_{r} \frac{\phi(u_r)}{\phi'(u_r)} = \frac{2x'S'}{4x'^3 - g_2x'^2 - g_3} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)
\]
which is a well-known formula.

In conclusion, I wish to thank my colleague Mr. M. Bhimasena Rao for helpful discussions and the Referee for suggesting a recasting of the paper.