ON THE MOTION OF A VARIABLE SPHERE OR CYLINDER IN A LIQUID

BY V. R. THIRUVENKATACHAR
(Department of Mathematics, Central College, Bangalore).

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§1. In this note we consider the motion through a liquid of a sphere (or cylinder) whose radius is variable. We expressly stipulate that the moving body retains its spherical (or cylindrical) shape throughout the motion. An example is furnished by the motion of a sphere on which material is deposited, or from which material is dissolved, uniformly, as it moves through the liquid. The problem becomes essentially more complicated in the case of a bubble of gas initially spherical in form, for the non-uniform distribution of pressure at the surface of the bubble, consequent on its motion, distorts the spherical shape of the bubble and the question arises as to the subsequent shape of the bubble for a given type of motion. This latter problem will not be considered here.

§2. Let R be the radius of the sphere at any time, and let the sphere move with velocity U in a straight line in a liquid at rest at infinity. Taking the origin at the centre of the moving sphere, the velocity-potential φ has to satisfy the conditions

(i) \( \nabla^2 \phi = 0 \)
(ii) \( \phi \rightarrow 0 \) as \( r \rightarrow \infty \)
(iii) \( -\left( \frac{\partial \phi}{\partial r} \right) = \dot{R} + U \cos \theta \), for \( r = R \).

These conditions are satisfied by taking

\[ \phi = \frac{R^2 \dot{R}}{r} + \frac{1}{2} \frac{UR^3}{r^2} \cos \theta \] (1)

the corresponding stream-function is easily verified to be

\[ \psi = R^2 \dot{R} \cos \theta - \frac{1}{2} \frac{UR^3}{r} \sin^2 \theta \] (2)

From (1) we find

\[ q^2 = \left( \frac{\partial \phi}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right)^2 \]

\[ = \frac{R^4 \ddot{R}^2}{r^4} + \frac{2R^5 \dot{R}U \cos \theta}{r^5} + \frac{R^6 U^2}{r^6} (\cos^2 \theta + \frac{1}{2} \sin^2 \theta) \]
and

\[ \frac{\partial \phi}{\partial t} = \frac{R^2 \ddot{R} + 2R \dot{R}^2}{r} + \frac{5}{2} \frac{R^2 \ddot{R} U \cos \theta}{r^2} + \frac{R^3 \cos \theta}{2r^2} \frac{dU}{dt} + \frac{R^3 U^2}{2r^3} (3 \cos^2 \theta - 1) \]

The pressure equation is

\[ \frac{p}{\rho} + \frac{1}{2} q^2 - \frac{\partial \phi}{\partial t} = F(t) \]

This gives

\[ \frac{p}{\rho} + \frac{1}{2} \left\{ \frac{R^4 \ddot{R}^2}{r^4} + \frac{2R^5 \ddot{R} U \cos \theta}{r^5} + \frac{R^6 U^2}{4r^6} (3 \cos^2 \theta + 1) \right\} \]

\[ - \frac{R^2 \ddot{R} + 2R \dot{R}^2}{r^5} - \frac{5}{2} \frac{\dot{R} R^2 U \cos \theta}{2r^2} + \frac{R^3 \cos \theta}{2r^2} \frac{dU}{dt} \]

\[ - \frac{R^3 U^2}{2r^3} (3 \cos^2 \theta - 1) = \frac{\Pi}{\rho} \tag{3} \]

where \( \Pi \) is the pressure at infinity.

Thus the pressure on the surface of the sphere is given by

\[ \frac{p - \Pi}{\rho} = R \ddot{R} + 2 \dot{R}^2 + \frac{5}{2} \dot{R} U \cos \theta + \frac{R}{2} \cos \theta \frac{dU}{dt} + \frac{U^2}{2} (3 \cos^2 \theta - 1) \]

\[ - \frac{1}{2} \left( \dot{R}^2 + 2 \dot{R} U \cos \theta + \frac{1}{2} U^2 (3 \cos^2 \theta + 1) \right) \]

\[ = R \ddot{R} + \frac{5}{2} \dot{R}^2 + \frac{3}{2} \dot{R} U \cos \theta + \frac{1}{2} R \cos \theta \frac{dU}{dt} \]

\[ + \frac{1}{2} U^2 (9 \cos^2 \theta - 5) \tag{4} \]

The kinetic energy of the liquid is given by

\[ T = - \pi \rho \int \phi \, d\psi = \pi \rho \int_0^{\pi} (R \ddot{R} + \frac{1}{2} R U \cos \theta) (R^2 \ddot{R} \sin \theta + R^2 U \sin \theta \cos \theta) \, d\theta \]

\[ = \pi \rho (2 \ddot{R} R^3 + \frac{1}{2} R^3 U^2) \]

\[ = \frac{1}{2} M' (3 \ddot{R}^2 + \frac{1}{2} U^2) \tag{5} \]

where \( M' \) is the mass of liquid displaced by the sphere.

If \( F \) is the resistance of the liquid, we have

\[ F U = \frac{dT}{dt} = M' \left( 3 \ddot{R} \dot{R} + \frac{1}{2} U \frac{dU}{dt} \right) \tag{6} \]

In particular if \( U = \text{const.} \), and \( R = a + A \sin nt \), we find from (4) the mean pressure on the sphere over a large time-interval is given by

\[ \frac{p_0 - \Pi}{\rho} = \pi a n^2 A^2 + \frac{1}{2} U^2 (9 \cos^2 \theta - 5) \tag{7} \]
§3. The corresponding problem in two dimensions may also be treated by the same method. We now assume for the complex potential

$$w = - R \bar{R} \log z + \frac{UR^2}{z}$$  \hspace{1cm} (8)

which gives, for the velocity-potential and stream-function

$$\phi = - R \bar{R} \log r + \frac{UR^2 \cos \theta}{r}$$

$$\psi = - R \bar{R} \theta - \frac{UR^2 \sin \theta}{r}$$  \hspace{1cm} (9)

To determine the force on the cylinder we make use of the extended form of Blasius' theorem (cf. Milne-Thomson, *Theoretical Hydrodynamics*, §9.52), which becomes, for the present case,

$$X - iY = \frac{1}{2} i \rho \int \left( \frac{d \bar{w}}{dz} \right)^2 dz - i \rho \frac{\partial}{\partial t} \int \bar{w} \, d\bar{z} + \pi \rho R^2 \frac{dU}{dt}$$  \hspace{1cm} (10)

Substituting for $w$ and $\bar{w}$ from (8) we find, after some reduction

$$X - iY = 2\pi \rho \left[ R^2 \bar{R} + 2R \bar{R}^2 - 2UR\bar{R} \right] - \pi \rho R^2 \frac{dU}{dt}$$

so that

$$Y = 0$$

$$X = 2\pi \rho \left[ R^2 \bar{R} + 2R \bar{R}^2 - 2UR\bar{R} \right] - \pi \rho R^2 \frac{dU}{dt}$$  \hspace{1cm} (11)

For $R =$ constant, this reduces to $X = - \pi \rho R^2 \frac{dU}{dt}$, a known result.