Linear Hypergraph Set-Indexers of a Graph

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Abstract
Given any graph $G = (V, E)$ and an arbitrary nonempty ‘ground set’ $X$, a linear hypergraph set-indexer (LHSI) is a set-valued function $f : V(G) \to 2^X$ assigning nonempty subsets of $X$ to the vertices of $G$ and satisfying the following conditions: (i) the ordered pair $H_f(G) = (X, f(V))$, where $f(V) = \{f(v) : v \in V(G)\}$, is a ‘simple’ hypergraph, in the sense that $f$ is injective and $\cup_{v \in V(G)} f(v) = X$, (ii) $H_f(G)$ is a linear hypergraph, in the sense that $|f(u) \cap f(v)| \leq 1 \forall u, v \in V$, (iii) the induced set-valued function $f_\oplus : E \to 2^X$ defined by $f_\oplus(uv) = f(u) \oplus f(v), \forall uv \in E$ is injective, and (iv) $H_{f_\oplus}(G) = (X, f_\oplus(\emptyset))$, where $f_\oplus(\emptyset) = \{f_\oplus(e) : e \in E\}$, is a linear hypergraph. In this paper, we initiate a study of linear hypergraph set-indexers of a graph.

Mathematics Subject Classification: 05C78
Keywords: Set-indexer, hypergraph set-indexer
1 Introduction

For all terminology and notation in graph theory and hypergraph theory, not specifically defined in this paper, we refer the reader to Harary [9] and Berge [6], respectively. Unless mentioned otherwise, all graphs considered in this paper are simple and without self-loops whereas hypergraphs are simple but may have loops (i.e., edges of cardinality one).

Let $X$ be a nonempty finite set, and let $E = \{E_i : i \in I\}$ be a family of subsets of $X$. The family $E$ is called a hypergraph on $X$ if (i) $E_i \neq \emptyset$, for every $i \in I$, and (ii) $\cup_{i \in I} E_i = X$; then, $H = (X, E)$ is called a hypergraph and $|X| = n$ is called its order. If all the edges of $H$ are distinct, then $H$ is said to be simple and $H$ is said to be linear if it satisfies the condition $|E_i \cap E_j| \leq 1$ for all $E_i, E_j \in E$ [6].

A set-valuation of a given graph $G$ is an assignment $f$ of subsets of an arbitrary nonempty 'ground set' $X$ to the vertices of $G$ and the symmetric difference $f^\oplus(uv) = f(u) \oplus f(v) := (f(u) - f(v)) \cup (f(v) - f(u))$ to each edge $uv \in E(G)$ [1,3,4,5,7]; this notion gave an altogether new direction to the topic of 'graph labeling' [8].

Next, for a simple graph $G = (V, E)$ and for an arbitrary set $X$, if $f : V(G) \rightarrow 2^X$, where $2^X$ is the set of all subsets of $X$, is a set-valuation such that $f(u) \neq \emptyset$ for each $u \in X$ and if $\cup_{v \in V(G)} f(v) = X$, then $H_f(G) = (X, f(V)), f(V) := \{f(v) : v \in V(G)\}$, is a hypergraph. Hence, given a property $\mathcal{P}$ of the subsets of $X$ a study of $\mathcal{P}$-hypergraphs associated with a given set-valuation of a given graph $G$ could be interesting. Often, specific properties $\mathcal{P}$ are suggested from practical contexts. In this paper, we consider studying this problem with $\mathcal{P}$ taken as the property of linearity of hypergraphs, as formulated below.

**Definition 1.1.** For a simple graph $G$, a set-valued function $f : V(G) \rightarrow 2^X$ is a linear hypergraph set-indexer (or, 'LHSI' in short) of $G$ if $f$ satisfies the following conditions: (i) $f$ is injective, (ii) $H_f(G)$ is a linear hypergraph, (iii) The induced set-valued function $f^\oplus : E \rightarrow 2^X$ defined by $f^\oplus(uv) = f(u) \oplus f(v)$, $\forall uv \in E$, is injective, and (iv) $H_{f^\oplus}(G) = (X, f^\oplus(E))$ is a linear hypergraph where, $f^\oplus(E) := \{f^\oplus(e) : e \in E\}$. The least (largest) cardinality of the set $X$ with respect to which $G$ admits an LHSI is called the linear hypergraph set-indexing ('LHSI', in short) number (upper LHSI number) of $G$, and it is denoted by $I_L(G)$ (respectively, $I^U_L(G)$).

2 Main Results

2.1 LHSI set-graceful and set-sequential graphs

For a $(p,q)$-graph $G = (V,E)$ and a nonempty set $X$, Acharya [1] defined a set-indexer of $G$ as an injective set-valuation (or, equivalently, 'set-labeling')
The graph $f : V(G) \rightarrow 2^X$ such that the induced ‘edge-function’ $f^\oplus : E(G) \rightarrow 2^X - \{\emptyset\}$ defined by $f^\oplus(uv) = f(u) \oplus f(v)$ for all $uv \in E(G)$ is also injective. A graph $G$, not necessarily finite, is called set-graceful if it admits a set-graceful labeling (or, graceful set-indexer) $f$, which is defined as a set-indexer such that $f^\oplus(E(G)) := \{f^\oplus(uv) : uv \in E(G)\} = 2^X - \{\emptyset\}$.

**Theorem 2.1.** A graph $G$ has a set-graceful LHSI if and only if $G \cong K_3$.

**Proof.** If $G \cong K_3$, let $V(K_3) = \{u_1, u_2, u_3\}$. Then, for $X = \{a, b\}$ it is easy to verify that the graceful set-indexer $f$ defined by $f(u_1) = \{a\}$, $f(u_2) = \{b\}$, and $f(u_3) = \{a, b\}$ is indeed a set-graceful LHSI of $K_3$. Conversely, let $G$ have a set-graceful LHSI $f : V(G) \rightarrow 2^X - \{\emptyset\}$. If $|X| = n \geq 3$ then let $X = \{x_1, x_2, \ldots, x_n\}$. Since $G$ is set-graceful and $f$ is a set-indexer of $G$, it follows that $f^\oplus(E) = 2^X - \{\emptyset\}$. But, then, $\{x_1, x_2\} \subset \{x_1, x_2, x_3\}$ whence for the edges $uv$ and $xy$ of $G$ with $f^\oplus(uv) = \{x_1, x_2\}$ and $f^\oplus(xy) = \{x_1, x_2, x_3\}$ we get $|f^\oplus(uv) \cap f^\oplus(xy)| = 2 > 1$, a contradiction to the definition of an LHSI. Therefore, $n \leq 2$. Now, since $|X| = 1$ is not possible, we have $n = 2$. Then, $X = \{x_1, x_2\}$ and since $G$ is set-graceful we must have $f^\oplus(E) = \{\{x_1\}, \{x_2\}, \{x_1, x_2\}\}$. The only set-labeling $f$ for which $f^\oplus(E)$ is as described is the set-indexer $f$ of $G = K_3$, $V(K_3) = \{a, b, c\}$, defined by $f(a) = \{x_1\}$, $f(b) = \{x_2\}$, and $f(c) = \{x_1, x_2\}$. This completes the proof. \qed

A graph $G$ is called set-sequential if $G$ admits a set-sequential labeling, which is a bijection $f : V \cup E \rightarrow 2^X - \{\emptyset\}$ such that $f(uv) = f(u) \oplus f(v)$ for all $uv \in E(G)$ (cf.: [2,4,5,7]).

**Example 2.2.** The set-sequential labeling of $K_2$ where $V(K_2) = \{v_1, v_2\}$, defined by $f(v_1) = \{1\}$, $f(v_2) = \{2\}$, $f^\oplus(v_1v_2) = \{1, 2\}$ is an LHSI of $K_2$. Further, notice that by augmenting a new vertex $v_3$ to $K_2$, with $\{1, 2\}$ as its set-label, and then by adding the new edges $v_3v_1$ and $v_3v_2$ we obtain a graceful LHSI of $K_3$ (also, see proof of Theorem 2.1), having the interesting property: $H_f(K_3) \cong H_{f^\oplus}(K_3)$. \qed

**Example 2.2** raises the following general problem.

**Problem 1.** Characterize graphs $G$ possessing an LHSI $f$ such that $H_f(G) \cong H_{f^\oplus}(G)$.

**Example 2.3.** Consider the 3-star $K_{1,3}$ with $V(K_{1,3}) = \{v_1, v_2, v_3, v_4\}$, where $v_1$ is the central vertex. Define $f(v_1) = \{1, 2, 3\}$, $f(v_2) = \{1\}$, $f(v_3) = \{2\}$, $f(v_4) = \{3\}$. Then, $f$ is an LHSI of $K_{1,3}$ which also happens to be a set-sequential labeling of $K_{1,3}$. \qed

If a set of cardinality 4 and all of its nonempty subsets are contained in $f(V \cup E)$ where $f$ is a set-sequential labeling of $G$, then at least one of the hypergraphs $H_f(G)$ and $H_{f^\oplus}(G)$ cannot be linear. This leads to the following observation.
Observation 2.4. Let $G = (V, E)$ be a graph and $f : V(G) \to 2^X - \{\emptyset\}$ be a set-sequential labeling of $G$. Then $f$ is an LHSI of $G$ only if $2 \leq |X| \leq 3$. □

More transparently, it is not difficult to prove the following result.

**Theorem 2.5.** If $G$ is a graph without isolated vertices, then $G$ has a set-sequential LHSI if and only if $G \in \{K_2, K_{1,3}\}$.

### 2.2 LHSI number of a graph

**Theorem 2.6.** If $G$ is a graph without isolated vertices, then $G$ admits an LHSI.

**Proof.** Let $G = (V, E)$ be a $(p, q)$-graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. Consider $X = \{1, 2, \ldots, p\}$ and define $f : V(G) \to 2^X$ as $f(v_i) = \{i\}$, for every $v_i \in V(G)$. Clearly $f^\oplus : E(G) \to 2^X$ is injective and $\bigcup_{e \in E(G)} f^\oplus(e) \subseteq X$. Let $k \in X$ and $v_k \in V(G)$. Since $G$ has no isolated vertex, there exists a vertex $v_k$ such that $e' = v_kv_s$ is an edge of $G$ and $f^\oplus(e') = f^\oplus(v_kv_s) = \{k, s\}$. Thus, $k \in \bigcup_{e \in E} f^\oplus(e)$, which implies $X \subseteq \bigcup_{e \in E} f^\oplus(e)$. Thus, $X = \bigcup_{e \in E} f^\oplus(e)$ and, hence, $f$ is an LHSI of $G$. □

**Remark 2.7.** Let $G$ be a $(p, q)$-graph without isolated vertices, then $I_L(G) \leq p$. Also, strict inequality holds in the case of $K_3$ with $I_L(K_3) = 2$.

**Theorem 2.8.** Let $G$ be a $(p, q)$-graph and $X$ be an arbitrary nonempty ground set with cardinality $n$. A necessary condition for $G$ to admit an LHSI $f : V(G) \to 2^X$ is that $\max(p, q) \leq \frac{n(n+1)}{2}$.

**Proof.** Let $|X| = n$ and $f : V(G) \to 2^X$ be an LHSI of $G$. Then, $|f^\oplus(e_i) \cap f^\oplus(e_j)| \leq 1$ for all $i \neq j$. Let $S_i$ be the collection of subsets of $X$ containing exactly $i$ elements and let $F$ be a maximal collection of subsets of $X$, such that $E_1, E_2 \in F$ implies $|E_1 \cap E_2| \leq 1$. Then $F$ contains no subsets of $X$ with cardinality greater than or equal to 3. For, if $\{a, b, c\} \in F$ then none of the subsets $\{a, b\}, \{b, c\}, \{a, c\}$ can be in $F$. Let $F' = F \cup \{\{a, b\}, \{b, c\}, \{a, c\}\} - \{\{a, b, c\}\}$. Then, $|F'| = |F| + 2$. Also, $E_1, E_2 \in F' \Rightarrow |E_1 \cap E_2| \leq 1$, contradiction to the maximality of $F$. Hence, $F \subseteq S_1 \cup S_2$. If $E_1, E_2 \in S_1 \cup S_2$, then $|E_1 \cap E_2| \leq 1$. Thus, $F = S_1 \cup S_2$. Hence, $|F| = |S_1| + |S_2| = \binom{n}{1} + \binom{n}{2}$.

Now, $f^\oplus(E)$ is a collection of subsets of $X$ such that no two members of the collection have more than one element in common. Hence, $|f^\oplus(E)| \leq \binom{n}{1} + \binom{n}{2}$, which implies $|E| \leq \binom{n}{1} + \binom{n}{2}$. That is, $q \leq n + \frac{n(n-1)}{2}$ and, hence, $2q \leq n(n+1)$. By a similar argument, considering the set labels assigned to the vertices of $G$, we get $2p \leq n(n+1)$ and the theorem follows. □

**Theorem 2.9.** The LHSI number of a complete graph with at least 2 vertices, is given by

$$I_L(K_n) = \begin{cases} n & \text{if } n \neq 3 \\ n-1 & \text{if } n = 3 \end{cases}$$
Proof. The result is obvious when $n = 2$ and for $n = 3$, $I_L(K_3) = 2$. Consider $K_n$, $n \geq 4$. By Remark 2.7, $I_L(K_n) \leq n$ and since the size of $K_n$ is $\binom{n}{2}$, by Theorem 2.8, we have $2\frac{n(n-1)}{2} \leq |X||X| + 1$, which in turn implies $n - 1 \leq |X|$. Now, we shall show that if $|X| = n - 1$, there exists no LHSI of $K_n$ with $X$ as the underlying set.

If possible, let $X = \{a_1, a_2, \ldots, a_{n-1}\}$ and $f : V(G) \to 2^X$ be an LHSI. Then, by definition, the set-valued function $f^\oplus : E(G) \to 2^X$ is injective and $(X, f^\oplus(E))$ is a linear hypergraph. Let $S_i$ be the collection of subsets of $X$ containing exactly $i$ elements. Then, $|S_1| + |S_2| = \binom{n}{2}$. Since $f^\oplus(E)$ is a collection of subsets of $X$ such that no two members of the collection have more than one element in common, the collection of such subsets of $X$ is maximum in number only when it is $S_1 \cup S_2$. Hence, $f^\oplus(E) = S_1 \cup S_2$. Clearly, $f(V) \subset S_1 \cup S_2$. Since $|S_1| = n - 1$, and $|V| = n$, there exists a vertex $u$ such that $f(u) \in S_2$. Without loss of generality, let $f(u) = \{a_1, a_2\}$. Then, $f(V) \cap [S_1 - \{a_1\}, \{a_2\}] = \emptyset$, otherwise an edge would receive its induced set label from $S_3$. Also, $\{a_1\} \in f^\oplus(E) \Rightarrow \{a_2\} \in f(V)$. Similarly, $\{a_2\} \in f^\oplus(E) \Rightarrow \{a_1\} \in f(V)$. Since $n \geq 4$, the remaining set labels assigned to the vertices should necessarily be taken from $S_2$ and each of them must contain at least one of the elements $a_1$ or $a_2$. Let $\{a_1, a_k\} \in f(V)$ for some $k$. Then $\{a_1, a_k\} \oplus \{a_2\} = \{a_1, a_2, a_k\} \in f^\oplus(E)$, a contradiction to $f^\oplus(E) = S_1 \cup S_2$. Now, let $\{a_2, a_s\} \in f(V)$ for some $s$. Then, $\{a_1\} \oplus \{a_2, a_s\} = \{a_1, a_2, a_s\} \in f^\oplus(E)$, again a similar contradiction. Thus, $I_L(K_n) \geq n$. Now, invoking the inequality $I_L(K_n) \leq n$, we get $I_L(K_n) = n$ for $n \geq 4$. 

Remark 2.10. For any graph $G$, $G \not\cong K_3$, that admits an LHSI we must have $I_L(G) \geq \omega(G)$, where $\omega(G)$ denotes the clique number of $G$. Hence, an interesting problem is to determine the bounds for the LHSI number.

Theorem 2.11. If $G$ is a $(p, q)$ graph without isolated vertices, then

$$\frac{1}{2} \sqrt{8 \max(p, q) + 1} - 1 \leq I_L(G) \leq p$$

Proof. We have $I_L(G) \leq p$. If $f : V(G) \to 2^X$ is an LHSI with $|X| = n$, by Theorem 2.8, $2p \leq n(n + 1)$. Hence, $2p + \frac{1}{2} \leq n^2 + n + \frac{1}{2} \Rightarrow \frac{8p + 1}{2} \leq (n + 1)^2$. On simplification, we get $n \geq \frac{1}{2}(\sqrt{8p + 1} - 1)$. Hence, $I_L(G) \geq \frac{1}{2}(\sqrt{8p + 1} - 1)$. Replacing $p$ by $q$ in the above argument, we get $I_L(G) \geq \frac{1}{2}(\sqrt{8q + 1} - 1)$ and, therefore, $\frac{1}{2}(\sqrt{8 \max(p, q) + 1} - 1) \leq I_L(G) \leq p$. 

Remark 2.12. The bounds for $I_L(G)$ given in Theorem 2.11 are sharp. The upper bound is attained by all complete graphs of order $\geq 4$. The lower bound for $I_L(G)$, cited in Theorem 2.11 is attained by the path $P_6$. Let $V(P_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and let $X = \{a, b, c\}$. The set-valued function
f : V(P_6) → 2^X defined by f(v_1) = \{a\}, f(v_2) = \{b\}, f(v_3) = \{b,c\}, f(v_4) = \{c\}, f(v_5) = \{a,c\}, f(v_6) = \{a,b\} is an LHSI of P_6 with |X| = 3. The lower bound for I_L(G) involving q is attained by the graph obtained by attaching one vertex adjacent to each vertex of the complete graph K_n, where n ≥ 4.

**Proposition 1.** The complete graph K_3 admits an LHSI \( f : V(K_3) → 2^X \) if and only if \(|X| = 2 \) or 3.

**Proof.** Let \( f : V(K_3) → 2^X \) be an LHSI. Let \( A_1, A_2, A_3 \) be the subsets of \( X \) assigned to the vertices. The induced set-valued function \( f^\oplus : E(K_3) → 2^X \) is injective and \((X, f^\oplus(E))\) is a linear hypergraph, which implies \(|(A_1 \oplus A_2) \cap (A_1 \oplus A_3)| ≤ 1 \). Hence,

\[
|(A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3^c)| ≤ 1. \tag{1}
\]

Since \( X = A_1 \cup A_2 \cup A_3 \), \( X \) can be considered as the union of seven disjoint sets \( A_1 \cap A_2^c \cap A_3^c, A_1^c \cap A_2 \cap A_3^c, A_1^c \cap A_2 \cap A_3, A_1 \cap A_2 \cap A_3^c, A_1 \cap A_2^c \cap A_3, A_1 \cap A_2^c \cap A_3^c \). The inequality (1) shows that the sets \( A_1 \cap A_2^c \cap A_3^c \) and \( A_1^c \cap A_2 \cap A_3^c \) together contain at most one element of \( X \). Similarly, interchanging the role of \( A_1 \) and \( A_2 \) in the inequality (1), the sets \( A_1^c \cap A_2 \cap A_3^c \) and \( A_1 \cap A_2^c \cap A_3 \) together contain at most one element of \( X \). Similar is the case for \( A_1^c \cap A_2^c \cap A_3^c \) and \( A_1 \cap A_2 \cap A_3 \). If \( x \in A_1 \cap A_2 \cap A_3 \), then \( x \) does not belong to \( \cup_{e \in E} f^\oplus(e) = X \). Hence, the maximum possible cardinality of \( X \) is 3. Also, \( K_3 \) admits an LHSI when \(|X| = 2 \) and \(|X| = 3 \).

For the complete graph \( K_3 \) to admit an LHSI, the cardinality of the underlying set cannot exceed 3. In general, we can see that \(|X|\) cannot exceed \( n \) for any complete graph \( K_n \), \( n ≥ 3 \), which is our next theorem.

**Theorem 2.13.** The complete graph \( K_n \), \( n ≥ 4 \), admits an LHSI with \( X \) as the underlying set, if and only if \(|X| = n \).

**Proof.** By Theorem 2.11, \( K_n \), \( n ≥ 4 \), admits an LHSI with \(|X| = n \). Now, let \( A_1, A_2, \ldots, A_n \) be the subsets of \( X \) assigned to the vertices of \( K_n \) under an LHSI \( f : V(K_n) → 2^X \). Then \( \cup_i A_i = X \). Since \( \cup_{e \in E} f^\oplus(e) = X, A_1 \cap A_2 \cap \cdots \cap A_n = \emptyset \). Since any 3 vertices of a complete graph lie on a triangle, invoking the arguments in the proof of Proposition 1, \(|A_i \cup A_j \cup A_k| ≤ 4 \), for all distinct indices \( i, j, k \). Also, \(|A_i \cup A_j \cup A_k| = 4 \) only if there exists an element \( x \in A_i \cap A_j \cap A_k \). Let \( A_i = \{x, a\}, A_j = \{x, b\}, A_k = \{x, c\} \), for some \( a, b, c \in X \). Consider a set \( A_s \) assigned to a vertex of \( G \), not containing \( x \). If \( d \in A_s \) \(-\{a, b, c\} \) then \( \{d, x\} \in (A_1 \oplus A_s) \cap (A_2 \oplus A_s) \), contradicting the linearity of \( H_{f^\oplus}(G) \), which implies \( A_s \subset \{a, b, c\} \). When \( A_s \) is any one of the seven nonempty subsets of \( \{a, b, c\} \), it leads to a contradiction to the linearity of \( H_{f^\oplus}(G) \). Therefore, \(|A_i \cup A_j \cup A_k| ≤ 3 \), for all distinct indices \( i, j, k \) and \(|A_i \cup A_j| ≥ 2 \) for all \( i \neq j \). Hence, \(|A_1 \cup A_2 \cup \cdots \cup A_n| ≤ n \), which implies \(|X| ≤ n \). Now, if \( n ≥ 4 \), by Theorem 2.9, \(|X| = n \).
An interesting question is to find the maximum number of isolated vertices, admissible in a graph admitting an LHSI. For the complete graph $K_n, n \geq 3$, it is answered by Theorem 2.14.

**Theorem 2.14.** The graph $K_n \cup \overline{K}_m, n \geq 3,$ admits an LHSI if and only if $m \leq \binom{n}{2}$.

**Proof.** Let $m \leq \binom{n}{2}$ and $X = \{1, 2, 3, \ldots, n\}$. Consider $K_n \cup \overline{K}_m$, with $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(\overline{K}_m) = \{u_1, u_2, \ldots, u_m\}$. Let $f(v_i) = \{i\}, i \in \{1, 2, 3, \ldots, n\}$, $f(u_1), f(u_2), f(u_3), \ldots, f(u_m)$, be the subsets of $X$ with cardinality 2 assigned arbitrarily but injectively. Then, $f : V(K_n \cup \overline{K}_m) \to 2^X$ is an LHSI on $K_n \cup \overline{K}_m$.

Conversely, let $K_n \cup \overline{K}_m$ has an LHSI, with $X$ as the underlying set. Let $A_1, A_2, \ldots, A_n$ be the sets assigned to the vertices in $K_n$. If $x \in X - (A_1 \cup A_2 \cup \cdots \cup A_n)$, then $x$ does not belong to $\cup_{\epsilon \in E} f^\epsilon(e) = X$. Hence, $A_1 \cup A_2 \cup \cdots \cup A_n = X$, which implies $|X| = n$. Hence, applying Theorem 2.8, the result follows. \hfill \square

If $G$ has a pendant vertex, then $G$ admits an LHSI, irrespective of the number of isolated vertices in it.

**Theorem 2.15.** If $G$ is a graph with a pendant vertex, then $G \cup \overline{K}_m$ admits an LHSI, for every $m \in N$.

**Proof.** Without loss of generality, suppose $G$ contains no isolated vertex and let $u$ be a pendant vertex of $G$. Consider $G' = G - u$. Let $V(G') = \{v_1, v_2, \ldots, v_k\}$ and $V(\overline{K}_m) = \{u_1, u_2, \ldots, u_m\}$. Consider $X = \{1, 2, 3, \ldots, k, k+1, \ldots, k+m\}$. Define $f : V(G \cup \overline{K}_m) \to 2^X$ as follows: $f(v_i) = \{i\}, i \in \{1, 2, 3, \ldots, k\}$, $f(u) = \{k + 1, k + 2, k + 3, \ldots, k + m\}$, $f(u_i) = \{k + i\}, i \in \{1, 2, \ldots, m\}$. Then, $f$ is an LHSI of $G \cup \overline{K}_m$. \hfill \square

**Theorem 2.16.** Let $G = (V, E)$ be a $(p, q)$-graph and let the $f : V \to 2^X$ be an LHSI of $G$. Let $u$ be any vertex of $G$ with its vertex degree $d(u) \geq 2$. Then, $|f(u)| \leq 3$.

**Proof.** Let $f : V(G) \to 2^X$ is an LHSI and let $f(u) = A_1$. Let $v, w$ be two vertices of $G$ adjacent to $u$. Assume $f(v) = A_2$, $f(w) = A_3$. Then, we have, $|A_1 \cap A_2| \leq 1$ and $|A_1 \cap A_3| \leq 1$. Since $H_f \circ (G)$ is linear, $|f^\bigoplus(wv)| \leq 1$. That is, $|(A_1 \oplus A_2) \cap (A_1 \oplus A_3)| \leq 1$, which implies, $|A_1 \cap A_2^c \cap A_3^c| \cup (A_1^c \cap A_2 \cap A_3^c)| \leq 1$, implies $|A_1 \cap A_2^c \cap A_3^c| \leq 1$. But, $A_1 \subseteq (A_1 \cap A_2^c \cap A_3^c) \cup (A_1 \cap A_2 \cup A_1 \cap A_3)$. Hence, $|A_1| \leq |A_1 \cap A_2^c \cap A_3^c| + |(A_1 \cap A_2)| + |(A_1 \cap A_3)|$. That is, $|A_1| \leq 1 + 1 + 1 = 3$. Hence, $|f(u)| \leq 3$. \hfill \square

**Theorem 2.17.** Let $G = (V, E)$ be a graph and let $f : V(G) \to 2^X$ be an LHSI of $G$. Let $u$ be any vertex of $G$ with $d(u) \geq 4$. Then, $|f(u)| \leq 2$. 

Proof. Let \(v_1, v_2, v_3, v_4\) be any four vertices adjacent to \(u\). Let \(f(u) = A_1\) and \(f(v_i) = B_i, \quad i = 1, 2, 3, 4\). Since \(d(u) \geq 2\), \(|f(u)| \leq 3\). We shall now show that \(|f(u)| = 3\) is not possible. If possible, let \(|f(u)| = 3\) and let \(f(u) = \{a, b, c\}\). Then \(|A_i \cap B_j| \leq 1, \quad i = 1, 2, 3, 4\), which implies \(|A_i \cap B_j| \geq 2, \quad i = 1, 2, 3, 4\), which in turn implies \(|A_i \oplus B_j| \geq 2, \quad i = 1, 2, 3, 4\). Each \(A_i \oplus B_j\) would contain one of the sets \(\{a, b\}; \{b, c\}; \{a, c\}\). Hence, at least two of \(A_i \oplus B_j\) would contain the same set of the collection. That is, for \(i \neq j\), \((A_i \oplus B_j) \cap (A_j \oplus B_j) \geq 2\), which is a contradiction to the linearity of \(H_{f \oplus}(G)\).

If a graph \(G\) has a pendant vertex, the cardinality of the underlying set \(X\), with respect to which \(G\) admits an LHSI, can be made arbitrarily large by increasing the cardinality of the set assigned to the pendant vertex.

**Theorem 2.18.** For a simple graph \(G\) admitting an LHSI \(f : V(G) \to 2^X\), \(|X|\) can be any arbitrary positive integer greater than \(I_L(G)\) if and only if \(G\) contains a pendant vertex.

*Proof.* Suppose \(G\) contains a pendant vertex \(u\) and let \(f : V(G) \to 2^X\) be an LHSI of \(G\) with \(X = \{1, 2, \ldots, l\}\) where \(l = I_L(G)\). Let \(k\) be any positive integer such that \(k \geq l\) and let \(X' = \{1, 2, 3, \ldots, l, l + 1, \ldots, k\}\). Define \(f' : V(G) \to 2^X'\) as \(f'(v) = f(v)\), for all \(v \in V(G) - \{u\}\) and \(f'(u) = f(u) \cup \{l + 1, l + 2, \ldots, k\}\). Clearly, \(f' : V(G) \to 2^X'\) is an LHSI with \(|X'| = k\).

To prove the converse, we establish its contrapositive statement. Let \(G\) be a graph without any pendant vertex. If an element \(x \in X\) appears only in the set-label of an isolated vertex and not on the set-label of any non-isolated vertex, then \(x \not\in \cup_{e \in E(G)} f_\oplus(e)\). Hence, the union of the sets assigned to the non-isolated vertices constitute the underlying set of an LHSI. Let \(G\) contain \(p\) non-isolated vertices and let \(f : V(G) \to 2^X\) be any LHSI of \(G\). Since \(G\) has no pendant vertices, \(d(v) \geq 2\), for any non-isolated vertex \(v\) of \(G\). By Theorem 2.16, \(|X| \leq 3p\). \(\square\)

### 2.3 Upper LHSI number of a graph

For a simple graph \(G\) admitting an LHSI, recall from Definition 1.1 that its **upper LHSI number**, denoted \(I_{UL}(G)\), is the maximum cardinality of a ground set with respect to which \(G\) admits an LHSI.

**Remark 2.19.** By Theorem 2.13 and Proposition 1, \(I_{UL}(K_n) = n\), for all \(n \geq 3\).

**Proposition 2.** If \(G = (V, E)\) is a \((p, q)\)-graph without pendant vertices and isolated vertices then \(I_{UL}(G) \leq 2p\).
Let $f : V(G) \to 2^X$ be an LHSI of $G$. Since all the vertices of $G$ are of degree $\geq 2$, by Theorem 2.16 we have $|f(u)| \leq 3$ for all $u \in V(G)$. If $|f(u)| = 3$, then, by a similar argument as that we used in the proof of Theorem 2.17, two elements in $f(u)$ occur one each in the sets assigned to two vertices adjacent to $u$. Since $\cup_{u \in V} f(u) = X$, $|X| \leq 3p - 2q = 2p$, which implies $I^{UL}(G) \leq 2p$. 

Now, we shall determine the exact values of $I^{UL}(C_n)$. By Proposition 1, $I^{UL}(C_3) = 3$. Considering the sets assigned to the vertices of $C_4$ with all possible cardinalities under an LHSI, it can be shown that $I^{UL}(C_4) = 6$.

**Theorem 2.20.** $I^{UL}(C_n) = 2n$, if $n \geq 5$.

**Proof.** We have $I^{UL}(C_n) \leq 2n$. We shall construct an LHSI $f : V(G) \to 2^X$ with $|X| = 2n$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let $X = \{a_1, a_2, \ldots, a_n, 1, 2, \ldots, n\}$. Define $f : V(C_n) \to 2^X$ by letting

\[
 f(v_1) = \{a_1, 1, n\}; \quad f(v_2) = \{a_2, 1, 2\}; \quad f(v_3) = \{a_3, 2, 3\}; \quad \ldots; \quad f(v_i) = \{a_i, i-1, i\}; \quad \ldots; \quad f(v_n) = \{a_n, n-1, n\}.
\]

Then, $f^\oplus(v_1v_2) = \{a_1, a_2, n, 2\}; \quad f^\oplus(v_2v_3) = \{a_2, a_3, 1, 3\}; \quad \ldots; \quad f^\oplus(v_nv_1) = \{a_1, a_n, n-1, 1\}$. It is easily seen that both $H_f(G)$ and $H_f(G)$ are linear hypergraphs.

**Theorem 2.21.** For a $(p, q)$-graph $G$ with the minimum vertex degree $\delta(G) \geq 3$, $I^{UL}(G) \leq \frac{3p}{2}$.

**Proof.** Let $f : V(G) \to 2^X$ be an LHSI on a $(p, q)$-graph $G$ with $d(u) \geq 3$ for all $u \in V(G)$. Hence, let $p_1, p_2, p_3$ be the number of vertices of $G$ such that the cardinalities of the subsets of $X$ assigned to the vertices are $1, 2, 3$, respectively. Then, $p_1 + p_2 + p_3 = p$. If $|f(u)| = 3$, then $|f(u) \cap f(v_i)| = 1$, where $v_i$ is any vertex adjacent to $u$. Also, $f(v_i) \cap f(v_j) = \emptyset$, where both $v_i$ and $v_j$ are adjacent to $u$. Hence, each element in the $p_3$-subsets of $X$ assigned to the vertices of $G$ occur at least twice in the sets assigned to the vertices. Similarly, if $|f(u)| = 2$, then $|f(u) \cap f(v_i)| = 1$ for at least two vertices $v_i$ adjacent to $u$. Now, $\sum_{u \in V} |f(u)| = p_1 + 2p_2 + 3p_3$ and $X = \cup_{u \in V} f(u)$. Therefore, $|X| \leq p_1 + \frac{2p_2 + 3p_3}{2} \Rightarrow |X| \leq p + \frac{p_3}{2}$, which is maximum when $p_3 = p$. That is, when all the sets assigned to the vertices of $G$ are of cardinality $3$. The maximum possible value of $|X|$ is then $\frac{3p}{2}$. Thus, the result follows.

**Remark 2.22.** Let $G$ be a $(p, q)$-graph with $d(u) = 3$, for all $u \in V(G)$. Let $f : V(G) \to 2^X$ be an LHSI with $|X| = \frac{3p}{2}$. Then, $|f(u)| = 3$ for all $u \in V(G)$ and each element in $X$ occurs exactly twice in the sets assigned to the vertices.

**Theorem 2.23.** Let $G$ be a 3-regular graph of order $p$. Then, $I^{UL}(G) = \frac{3p}{2}$, if and only if $G$ contains no cycles of length $\leq 4$. 
Proof. Suppose \( I^{UL}(G) = \frac{3p}{2} \). Let \( f : V(G) \rightarrow 2^X \) be an LHSI of \( G \) with \(|X| = \frac{3p}{2}\). Now, \( G \) contains no triangle, since by Remark 2.22, \(|f(u)| = 3\) for all \( u \in V(G) \) and in particular, each set assignment of vertices of the triangle is of cardinality three. This leads to a contradiction to the linearity of \( H_{f^\oplus}(G) \), as argued in the proof of Proposition 1. Now, let \( G \) contain a cycle of length 4 and let \( A_1, B_1, A_2, B_2 \) be the sets assigned to the vertices \( v_1, v_2, v_3, v_4 \) in a cyclic order, under the LHSI \( f \). Invoking Remark 2.22, there exists an element \( x_1 \in A_1 \cap B_1 \) and \( x_1 \not\in A_2 \cup B_2 \). Similarly, there exists an element \( x_2 \in A_2 \cap B_2 \) and \( x_2 \not\in A_1 \cup B_1 \). Then, \( \{x_1, x_2\} \subseteq (A_1 \oplus B_2) \cap (A_2 \oplus B_1) \), contradicting the linearity of \( H_{f^\oplus}(G) \).

Conversely, let \( G \) be a 3-regular simple graph with no cycles of length \( \leq 4 \). By Theorem 2.21, \( I^{UL}(G) \leq \frac{3p}{2} \). Let \( X = \{e_1, e_2, \ldots, e_q\} \) be the set of edges, where \( q = \frac{3p}{2} \). Define \( f : V(G) \rightarrow 2^X \) as \( f(v) = \) set of edges incident with the vertex \( v \), for every \( v \in V(G) \). Clearly, \( f \) is injective and \( H_f(G) \) is linear. Then, the induced set-valuation \( f^\oplus : E(G) \rightarrow 2^X \) is given by \( f^\oplus(uv) = \) set of edges incident with exactly one of \( u \) and \( v \). Clearly, \( f^\oplus \) is injective.

**Claim:** \( f^\oplus : E(G) \rightarrow 2^X \) is linear.

Suppose \(|f^\oplus(u_1v_1) \cap f^\oplus(u_2v_2)| \geq 2\), where \( \{u_1, v_1\} \not= \{u_2, v_2\} \). Let \( e_1, e_2 \in f^\oplus(u_1v_1) \cap f^\oplus(u_2v_2) = S \), say.

**Case 1:** The edges \( u_1v_1 \) and \( u_2v_2 \) are adjacent. Without loss of generality, let \( v_1 = v_2 = v \). Then \( u_1v \) and \( u_2v \) are edges in \( G \) not belonging to \( S \). The third edge incident with \( v \) belongs to \( S \). Since \(|S| \geq 2\), there exists an edge \( e_i \in S \) which is not incident with \( v \). Then \( e_i = u_1u_2 \), whence \( u_1, u_2, v \) form a triangle in \( G \), a contradiction.

**Case 2:** The edges \( u_1v_1 \) and \( u_2v_2 \) are non-adjacent.

**Subcase 1:** Let \( e_1, e_2 \) be adjacent with \( u_1 \) as their common vertex. Then \( u_1, u_2, v_2 \) form a triangle in \( G \), a contradiction. **Subcase 2:** The edges \( e_1, e_2 \) are non-adjacent. Each of \( e_1 \) and \( e_2 \) has one end vertex in \( \{u_1, v_1\} \) and the other in \( \{u_2, v_2\} \). Therefore, \( u_1, v_1, u_2, v_2 \) are the vertices of a cycle of length 4 in \( G \), a contradiction. Thus, \( f : V(G) \rightarrow 2^X \) is an LHSI of \( G \) with \(|X| = \frac{3p}{2}\). Hence, \( I^{UL}(G) = \frac{3p}{2} \).

**Theorem 2.24.** Let \( G \) be a 3-regular \((p, q)\)-graph without cycles of length \( \leq 4 \). Then, there exists an LHSI \( f : V(G) \rightarrow 2^E(G) \) such that

1. \( G \) is isomorphic to the representative graph of \( H_f(G) \). (cf.: [6], p.400)

2. The line graph \( L(G) \) of \( G \) is isomorphic to a spanning subgraph of the representative graph of \( H_{f^\oplus}(G) \).

**Proof.** Let \( V(G) = \{v_1, v_2, \ldots, v_p\} \) and \( E(G) = \{e_1, e_2, \ldots, e_q\} \) where \( G \) is a 3-regular graph without cycles of length \( \leq 4 \). Define \( f(v) = E_v \), the set of edges incident with the vertex \( v \), for all \( v \in V(G) \). Then, the induced function \( f^\oplus : E(G) \rightarrow 2^X \) is given by \( f^\oplus(e_i) = \) set of all edges adjacent with \( e_i \) in \( G \). By
similar arguments in the proof of Theorem 2.23, \( f : V(G) \to 2^{E(G)} \) is an LHSI of \( G \). Two vertices \( v_i \) and \( v_j \) in \( G \) are adjacent if and only if there exists an edge \( e_k \) incident with both \( v_i \) and \( v_j \), which is true if and only if \( e_k \in f(v_i) \cap f(v_j) \), which, in turn, is true if and only if the vertices in the representative graph of \( H_f(G) \) corresponding to the sets \( f(v_i) \) and \( f(v_j) \) are adjacent. Thus, \( G \) is isomorphic to the representative graph of \( H_f(G) \), establishing part 1 of the conclusion of the theorem.

Next, the number of vertices in \( L(G) = \) the number of vertices in the representative graph of \( H_f \oplus (G) = q \). Let \( e_i' \) denote the vertex in \( L(G) \) corresponding to the edge \( e_i \) in \( G \) and let \( x_i \) denote the vertex in the representative graph \( L(H_f \oplus (G)) \) corresponding to the edge \( f \oplus (e_i) \) in \( H_f \oplus (G) \). The vertices \( e_i' \) and \( e_j' \) are adjacent in \( L(G) \) implies, the edges \( e_i \) and \( e_j \) in \( G \) are incident at a common vertex \( v_k \), say. That is, \( e_i, e_j \in f \oplus (e_k) \) where \( e_k \) is the third edge incident with \( v_k \). This implies, \( e_k \in f \oplus (e_i) \cap f \oplus (e_j) \), which, in turn implies, \( x_i \) and \( x_j \) are adjacent in \( L(H_f \oplus (G)) \).

\[ \square \]

3 Some new questions and an application

As we have seen, any graph \( G \) without isolated vertices admits an LHSI and also \( K_n \cup \overline{K}_m \) does not possess an LHSI when \( m \geq \binom{n}{2} \). Hence, it is the number of isolated vertices in a graph which determines whether the graph admits an LHSI. The following are some interesting problems which are under investigation.

**Problem 2.** Find the maximum number of isolated vertices admissible in a graph for it to possess an LHSI.

**Problem 3.** Find the possible relations between the given graph and the hypergraphs associated with a given set-indexer of the graph, with regard to their structural aspects like conformality and parameters like the coloring numbers, stability number, domination numbers, cyclomatic numbers, etc.

Linear hypergraph set-indexers of a graph have scope for application in game theory. For instance, let \( X \) be a group of players. They are to be divided into \( p \) different teams such that no two teams contain more than one member in common and some competitions, say \( q \) in number, are to be conducted between certain pairs of teams. When two teams compete, the common member if any, must stand aside without participation. Also, all players should take part in at least one competition and no two players should compete each other in more than one contest. With a given number of players, can we form \( p \) teams so that \( q \) assigned competitions are to be conducted between the teams? The problem reduces exactly to the problem of finding whether there exists an LHSI of the \((p,q)\)-graph with \( X \) as the underlying set, where vertices stand for
different teams and edges denote paired competitions. The study can further be extended to hypergraphs other than linear hypergraphs.

Acknowledgements
The authors thank the referees and Professor Thomas Zaslavsky for their critical comments and suggestions to improve presentation and clarity in the paper. The third author is indebted to the University Grants Commission (UGC) for granting her Teacher Fellowship (TF) under UGC’s Faculty Development Programme.

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Received: June, 2010