On a $_2\psi_2$ Basic Bilateral Hypergeometric Series Summation Formula

K. R. Vasuki

Department of Studies in Mathematics, University of Mysore
Manasagangotri, Mysore-570 006, India
vasuki.kr@hotmail.com

K. R. Rajanna

Department of Mathematics, MVJ College of Engineering
Channasandra, Bangalore-560 067, India
rajukarp@yahoo.com

Abstract

In this paper, we give an alternating proof of $_2\psi_2$ summation formula due to S. Bhargava and C. Adiga. Further, we deduce certain Rogers-Ramanujan type identities.

Mathematic Subject Classification: 33D15, 33D90

Keywords: Bilateral Basic Hypergeometric series, basic hypergeometric series, Rogers-Ramanujan type identities

1 Introduction

Throughout this paper, let $|q| < 1$. We adopt the following notation and terminology in [5]. As usual for any complex number $a$, we write

\[(a)_n := \begin{cases} 1, & \text{if } n = 0 \\ (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), & \text{if } n \geq 1 \end{cases},\]

and

\[(a)_\infty := \prod_{n=0}^{\infty} (1-aq^n).\]
Using the fact that \((a)_n = (a)_\infty / (aq^n)_\infty\), one can deduce that

\[
(a)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{a^n(q/a)_n}.
\]  

The basic hypergeometric series \(r+1 \varphi_r\) and the bilateral basic hypergeometric series \(r \psi_r\) are given by

\[
\begin{align*}
(r+1 \varphi_r) [a_1, a_2, \ldots, a_{r+1} ; b_1, b_2, \ldots, b_r ; z] & := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{r+1})_n}{(b_1)_n (b_2)_n \cdots (b_r)_n} z^n, \\
(r \psi_r) [a_1, a_2, \ldots, a_r ; b_1, b_2, \ldots, b_r ; z] & := \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_r)_n} z^n
\end{align*}
\]

respectively. One of the most important summation theorem for basic hypergeometric series is the Heine’s \(q\)-analog of the Gauss summation theorem[12]:

\[
2 \varphi_1 \left[ \frac{a}{b} ; \frac{c}{ab} \right] = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}, \quad |c/ab| < 1.
\]  

Similarly, one of the most interesting summation theorem for bilateral basic hypergeometric series is the following Ramanujan \(1 \psi_1\) summation theorem [18]:

\[
1 \psi_1 \left[ \frac{a}{b} ; z \right] = \frac{(az)_\infty (q/az)_\infty (b/a)_\infty (q)_\infty}{(z)_\infty (b/az)_\infty (q/a)_\infty (b)_\infty}, \quad |b/a| < |z| < 1.
\]  

The identity (1.3) was first brought before the mathematical world by G. H. Hardy [11, pp. 222, 223] who described it as “a remarkable formula with many parameters.” Hardy did not supply a proof but indicated that a proof could be constructed from the \(q\)-binomial theorem. The first published proof of (1.3) appears to be by W. Hahn [10] and M. Jackson [14]. Other proofs have been given by G. E. Andrews [1], [2], Andrews and R. Askey [3], Askey [4], M. E. H. Ismail [13], N. J. Fine [9], K. Mimachi [17], K. Venkatachaliengar [20], S. Corteel and J. Lovejoy [8], A. J. Yee [21], S. H. Chan [7], Z. G. Liu [16], K. W. J. Kadell [15].

S. Bhargava and C. Adiga [6] obtained the following \(2 \psi_2\) summation formula:
Hypergeometric series summation formula

\[ 2\psi_2 \left[ \frac{q/a, b}{d, bq}; a \right] = \frac{(d/b)_\infty (ab)_\infty (q)_\infty^2}{(q/b)_\infty (d)_\infty (a)_\infty (bq)_\infty}, \quad |a| < 1, |d| < 1. \quad (1.4) \]

In fact Bhargava and Adiga [6] proved (1.4) by method of Ismail [13] proof of (1.3) and demonstrated its diverse uses leading to sums of squares theorems, Ramanujan’s Fourier series developments related to theta functions, Lambert series identities related to Dedekind eta functions and \( q \)-gamma and \( q \)-beta identities.

Motivated by these, in Section 2 of this paper, we give an alternating proof of (1.4), by employing (1.2) and (1.3). Furthermore in Section 3, we deduce certain Rogers-Ramanujan type identities [5, p.77].

2 Proof of (1.4)

Theorem 2.1. The identity (1.4) holds for \(|a| < 1\) and \(|d| < 1\).

Proof. It is easy to see that

\[
2\psi_2 \left[ \frac{a, b}{c, dq}; z \right] - a \, 2\psi_2 \left[ \frac{a, b}{c, dq}; zq \right] = \sum_{n=-\infty}^{\infty} \frac{(an)(bn)n}{(cn)(dn)n} z^n (1 - aq^n)
\]

\[= \frac{(1 - (c/q))(1 - d)}{z(1 - (b/q))} \, 2\psi_2 \left[ \frac{a, b/q}{c/q, d}; z \right]. \quad (2.1)\]

Thus,

\[
2\psi_2 \left[ \frac{a, b}{c, dq}; z \right] - a \, 2\psi_2 \left[ \frac{a, b}{c, dq}; zq \right] = \frac{(1 - (c/q))(1 - d)}{z(1 - (b/q))} \, 2\psi_2 \left[ \frac{a, b/q}{c/q, d}; z \right].
\]

Also,

\[ a \, 2\psi_2 \left[ \frac{a, b}{c, dq}; zq \right] + \frac{a}{d}(1 - d) \, 2\psi_2 \left[ \frac{a, b}{c, d}; z \right] \]
\[
= \sum_{n=-\infty}^{\infty} \frac{(a_n b_n)_{n}}{(c_n d_{n})_{n-1}} z^n \left[ a q^n + \frac{a}{d} \right]
= \frac{a}{d} 2^{\psi_2} \left[ a, b \quad c, d q \ ; z \right].
\]

Thus,
\[
= \frac{a}{d} 2^{\psi_2} \left[ a, b \quad c, d q \ ; z \right].
\]

Adding (2.1) and the above, we obtain
\[
(1 - \frac{a}{d}) 2^{\psi_2} \left[ a, b \quad c, d q \ ; z \right] + \frac{a}{d} (1 - d) 2^{\psi_2} \left[ a, b \quad c, d \ ; z \right] = \frac{a}{d} 2^{\psi_2} \left[ a, b \quad c, d q \ ; z \right].
\]

Changing a to \(q/a\), c to \(bq\) and z to a in the above, we find that
\[
(1 - \frac{q}{ad}) 2^{\psi_2} \left[ \frac{q}{a}, b \quad c, d q \ ; a \right] + \frac{q}{ad} (1 - d) 2^{\psi_2} \left[ \frac{q}{a}, b \quad c, d q \ ; a \right] = \frac{a}{d} 2^{\psi_2} \left[ \frac{q}{a}, b \quad c, d \ ; a \right].
\]
Ramanujan’s \( _1\psi_1 \) summation formula (1.3), we deduce that

\[
_2\psi_2\left[ \frac{q/a, b/q}{b, d}; a \right] = \frac{(1 - (b/q))(ab - q)}{(b - d)(1 - b)} \, _2\psi_2\left[ \frac{q/a, b}{bq, d}; a \right].
\]

Using this in (2.2), we obtain

\[
(1 - (q/ad)) \, _2\psi_2\left[ \frac{q/a, b}{bq, dq}; a \right] = \frac{b(1 - d)(ad - q)}{ad(b - d)} \, _2\psi_2\left[ \frac{q/a, b}{bq, d}; a \right]. \tag{2.3}
\]

Let

\[
f(d) := \, _2\psi_2\left[ \frac{q/a, b}{bq, d}; a \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{(q/a)_n(b)_n}{(bq)_n(d)_n} a^n + \sum_{n=1}^{\infty} \frac{(1/b)_n(q/d)_n}{(a)_n(q/b)_n} d^n,
\]

upon using (1.1). The two series are convergent, when \( |a| < 1 \) and \( |d| < 1 \). As a function of \( d \), \( f(d) \) is clearly analytic for \( |d| < 1 \), when \( |a| < 1 \). From (2.3), we have

\[
f(d) = \frac{(1 - (d/b))}{(1 - d)} f(dq).
\]

Iterating the above \( n - 1 \) times, we get

\[
f(d) = \frac{(d/b)_n}{(d)_n} f(dq^n). \tag{2.4}
\]

Since \( f(d) \) is analytic for \( |d| < 1, |a| < 1 \) by letting \( n \to \infty \), we obtain

\[
f(d) = \frac{(d/b)_\infty}{(d)_\infty} f(0). \tag{2.5}
\]

However, since

\[
f(q) = \sum_{n=0}^{\infty} \frac{(q/a)_n(b)_n}{(bq)_n(q)_n} a^n
\]
by (1.2). Now setting $d = q$ in (2.5) and using the above, we obtain

$$f(0) = \frac{(ab)_\infty(q)_\infty^2}{(q/b)_\infty(bq)_\infty(a)_\infty}.$$  

Using this in (2.5), we obtain

$$f(d) = \frac{(d/b)_\infty(ab)_\infty(q)_\infty^2}{(d)_\infty(q/b)_\infty(bq)_\infty(a)_\infty}.$$  

This completes the proof.

3 Rogers-Ramanujan type identities.

Theorem 3.1. We have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 - bq^n} = \frac{(q)_\infty^2}{(b)_\infty(q/b)_\infty} \quad (3.1)$$

Proof. Letting $a \to \infty$ and then $d \to 0$ in (1.4), we deduce (3.1).

Corollary 3.2. We have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(n+1)}}{1 - q^{4n+1}} = \frac{(q^4; q^4)_\infty^2}{(q^4)_\infty(q^3; q^4)_\infty} \quad (3.2)$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(n+1)}}{1 - q^{4n+2}} = \frac{(q^4; q^4)_\infty^2}{(q^2; q^4)_\infty^2} \quad (3.3)$$

Proof. Changing $q$ to $q^4$ in (3.1) and then setting $b = q$ and $q^2$ respectively, we deduce (3.2) and (3.3).

Corollary 3.3. We have
\[ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)/2}}{1 - q^{5n+1}} = \frac{(q^5;q^5)_\infty^2}{(q^4;q^5)_\infty^2(q^5;q^5)_\infty}, \tag{3.4} \]

and

\[ \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)/2}}{1 - q^{5n+2}} = \frac{(q^5;q^5)_\infty^2}{(q^2;q^5)_\infty^2(q^2;q^5)_\infty}. \tag{3.5} \]

**Proof.** Changing \( q \) to \( q^5 \) in (3.1) and then substituting \( b=q \) and \( q^2 \) respectively we obtain (3.4) and (3.5).

Thus (3.1) can be employed to obtain numerous new identities of Rogers-Ramanujan type.

**Acknowledgement**

The first author is thankful to DST, New Delhi for awarding research project [No. SR/S4/MS:517/08] under which this work has been done.

**References**


[12] E. Heine, Untersuchungen über die Reihe

$$1 + \frac{(1 - q^\alpha)(1 - q^\beta)}{(1 - q)(1 - q^r)} x + \frac{(1 - q^\alpha)(1 - q^{\alpha+1})(1 - q^\beta)(1 - q^{\beta+1})}{(1 - q)(1 - q^2)(1 - q^r)(1 - q^{r+1})} x^2 + \cdots$$


[16] Z.-G. Liu, A proof of Ramanujan’s $\psi_1$ summation, preprint.


Received: June, 2009