Bounds on the Largest of Minimum Degree Laplacian Eigenvalues of a Graph

Chandrashekar Adiga and C. S. Shivakumar Swamy

(Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore 570 006, India)

E-mail: c.adiga@hotmail.com, cskswamy@gmail.com

Abstract: In this paper we give three upper bounds for the largest of minimum degree Laplacian eigenvalues of a graph and also obtain a lower bound for the same.

Key Words: Minimum degree matrix, minimum degree Laplacian eigenvalues.

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§1. Introduction

Let $G = (V,E)$ be a simple, connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Assume that the vertices are ordered such that $d_1 \geq d_2 \geq \ldots \geq d_n$, where $d_i$ is the degree of $v_i$ for $i = 1, 2, \ldots, n$. The energy of $G$ was first defined by I. Gutman [5] in 1978 as the sum of the absolute values of its eigenvalues. The energy of a graph has close links to Chemistry (see for instance [6]). The $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of $G$, where

$$d_{ij} = \begin{cases} 
\min\{d_i, d_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\
0 & \text{otherwise.}
\end{cases}$$

This was introduced and studied in [1]. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\phi(G; \lambda) = \det(\lambda I - m(G)) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-1}\lambda + c_n,$$  \hspace{1cm} (1.1)

where $I$ is the unit matrix of order $n$. The minimum degree Laplacian matrix of $G$ is $L(G) = D(G) - m(G)$, where $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$. $L(G)$ is a real, symmetric matrix. The minimum degree Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph $G$, assumed in the non increasing order, are the eigenvalues of $L(G)$. The Laplacian matrix of $G$ is $L_1(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of $G$. The eigenvalues of the laplacian matrix $L_1(G)$ are important in graph theory, because they have relations to numerous graph invariants including connectivity, expanding property, isoperimetric number, independence number, genus, diameter, mean distance, and bandwidth-type parameters of a graph (see, for example, [2,3,9,10]). In

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many applications one needs good bounds for the largest Laplacian eigenvalue (see for instance [2,3,9,10]). In this paper, we give three upper bounds and a lower bound for $\mu_1$ the largest of minimum degree Laplacian eigenvalues of a graph.

§2. Main Results

In this section, we will give three upper bounds for $\mu_1$ the largest of minimum degree Laplacian eigenvalues of a graph. We employ the following theorem to prove one of our main results.

Theorem 2.1 ([4]) Let $G$ be a simple graph with $n$ vertices and $m$ edges, and let $\Pi = (d_1, d_2, \ldots, d_n)$ be the degree sequence of $G$. Then,

$$d_1^2 + d_2^2 + \ldots + d_n^2 \leq m\left(\frac{2m}{n-1} + n - 2\right).$$

Theorem 2.2 Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$\mu_1 \leq \frac{2m + \sqrt{(n-1)\left[n(2|c_2| + m\left(\frac{2m}{n-1} + n - 2\right) - 4m^2\right)}}}{n},$$

where $c_2$ is the coefficient of $\lambda^{n-2}$ in $\det(\lambda I - m(G))$.

Proof Clearly

$$\mu_1 + \mu_2 + \ldots + \mu_n = \text{Trace}[L(G)] = \sum_{v \in V(G)} d_v, \quad (2.1)$$

$$\mu_1^2 + \mu_2^2 + \ldots + \mu_n^2 = 2|c_2| + \sum_{i=1}^{n} d_i^2. \quad (2.2)$$

By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \left(\sum_{i=1}^{n} b_i^2\right). \quad (2.3)$$

Putting $a_i = 1$ and $b_i = \mu_i$ for $i = 2, \ldots, n$ in (2.3), we get

$$\left(\sum_{i=1}^{n} \mu_i - \mu_1\right)^2 \leq (n-1) \left(\sum_{i=1}^{n} \mu_i^2 - \mu_1^2\right).$$

Using (2.1) and (2.2) in above inequality, we obtain

$$\left(\sum_{v \in V(G)} d_v - \mu_1\right)^2 \leq (n-1) \left[2|c_2| + \sum_{i=1}^{n} d_i^2\right] - (n-1)\mu_1^2.$$
i.e., \[ n\mu_1 - \sum_{v \in V(G)} d_v \leq \sqrt{(n-1) \left[ n(2|c_2| + \sum_{i=1}^n d_i^2) - \left( \sum_{i=1}^n d_i \right)^2 \right]}. \]

Therefore

\[ \mu_1 \leq \frac{\sum_{i=1}^n d_i + \sqrt{(n-1) \left[ n \left( 2|c_2| + \sum_{i=1}^n d_i^2 \right) - \left( \sum_{i=1}^n d_i \right)^2 \right]}}{n}. \tag{2.4} \]

Employing Theorem 2.1 and \( \sum_{i=1}^n d_i = 2m \) in (2.4), we see that

\[ \mu_1 \leq \frac{2m + \sqrt{(n-1) \left[ n(2|c_2| + m \left( \frac{2m}{n-1} + n - 2 \right) - 4m^2 \right]}}{n}. \]

This completes the proof. \( \square \)

The following theorem gives another type of upper bound for \( \mu_1 \).

**Theorem 2.3** Let \( G \) be connected graph with \( n \) vertices and \( m \) edges. Then

\[ \mu_1 \leq \sqrt{2d_1^2 + 4m - 2d_1^2(n - d_1)}. \]

**Proof** Suppose that \( X = (x_1, x_2, x_3, \ldots, x_n)^T \) be an eigenvector with unit length corresponding to \( \mu_1 \). Then

\[ L(G)X = \mu_1 X. \]

Hence, for \( u \in V(G) \),

\[ \mu_1 x_u = d_u x_u - \sum_{v \in V(G)} d_{uv} x_v. \]

Here \( x_u \) we mean \( x_i \) if \( u = v_i \). Therefore

\[ \mu_1 x_u = \sum_{vu \in E(G)} (x_u - \min(d_u, d_v)x_v). \tag{2.5} \]

By Cauchy-Schwarz inequality, we have

\[ \mu_1^2 x_u^2 \leq \left( \sum_{vu \in E(G)} 1^2 \right) \left( \sum_{vu \in E(G)} (x_u - \min(d_u, d_v)x_v)^2 \right) \]

\[ \quad = d_u \left[ \sum_{vu \in E(G)} x_u^2 + \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2 - 2x_u \min(d_u, d_v) x_v \right]. \]

Observe that

\[ -2x_u \sum_{vu \in E(G)} \min(d_u, d_v)x_v \leq d_u x_u^2 + \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2. \tag{2.6} \]
Hence,

\[
\mu_1^2 x_u^2 \leq d_u \left[ \sum_{vu \in E(G)} x_u^2 + \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2 + d_u x_u^2 + \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2 \right].
\]

i.e.,

\[
\mu_1^2 x_u^2 \leq 2d_u^2 x_u^2 + 2d_u \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2. 
\]

Consequently,

\[
\mu_1^2 = \mu_1^2 \sum_{u \in V(G)} x_u^2 \\ \leq \sum_{u \in V(G)} [2d_u^2 x_u^2 + 2d_u \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2] \\ = 2 \sum_{u \in V(G)} d_u^2 x_u^2 + 2 \sum_{u \in V(G)} d_u \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2.
\]

Thus

\[
\mu_1^2 \leq 2d_1^2 + 2 \sum_{u \in V(G)} d_u \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2. 
\]

Now let \( v \sim u \) mean that \( u \) and \( v \) are not adjacent. Then

\[
\sum_{u \in V(G)} d_u \sum_{vu \in E(G)} \min(d_u, d_v)^2 x_v^2 \\ = \sum_{u \in V(G)} d_u \left[ 1 - \sum_{v \sim u} \min(d_u, d_v)^2 x_v^2 \right] = 2m - \sum_{u \in V(G)} d_u \sum_{v \sim u} \min(d_u, d_v)^2 x_v^2 \\ = 2m - \left( \sum_{u \in V(G)} d_u \min(d_u, d_v)^2 x_v^2 + \sum_{u \in V(G)} d_u \sum_{v \sim u, v \neq u} \min(d_u, d_v)^2 x_v^2 \right) \\ \leq 2m - \left( d_n^2 \sum_{u \in V(G)} d_u x_u^2 + \sum_{u \in V(G)} d_n \sum_{v \sim u, v \neq u} d_n^2 x_v^2 \right) \\ = 2m - \left( d_n^2 \sum_{u \in V(G)} d_u x_u^2 + \sum_{u \in V(G)} d_n^3 (n - d_u - 1)x_u^2 \right) \\ = 2m - \left( d_n^2 \sum_{u \in V(G)} d_u x_u^2 + \sum_{u \in V(G)} d_n^3 nx_u^2 - d_n^3 \sum_{u \in V(G)} d_u x_u^2 - d_n^3 \sum_{u \in V(G)} x_u^2 \right) \\ \leq 2m - d_n^3 \sum_{u \in V(G)} (n - d_1)x_u^2 \\ = 2m - d_n^3 (n - d_1).
\]

Hence, employing this in (2.8) we have

\[
\mu_1^2 \leq 2d_1^2 + 4m - 2d_n^3 (n - d_1).
\]
Therefore
\[ \mu_1 \leq \sqrt{2d_1^2 + 4m - 2d_1^2(n - d)}. \]

**Theorem 2.4** Let \( G \) be a connected graph then
\[ \mu_1 \leq \max \left( \sqrt{2(d_u^2 + d_u^2m_u d_u)} : u \in V(G) \right). \]

**Proof** From (2.7) we have
\[ \mu_1^2 x_u^2 \leq 2d_u^2x_u^2 + 2d_u \sum_{v : vu \in E(G)} \min(d_u, d_v)^2 x_v^2. \]

Thus
\[ \mu_1^2 \sum_{u \in V(G)} x_u^2 \leq 2 \sum_{u \in V(G)} d_u^2x_u^2 + 2 \sum_{u \in V(G)} d_u \sum_{v : vu \in E(G)} \min(d_u, d_v)^2 x_v^2. \]
\[ \leq 2 \sum_{u \in V(G)} d_u^2x_u^2 + 2d_1^2 \sum_{u \in V(G)} d_u \sum_{v : vu \in E(G)} x_v^2 \]
\[ = 2 \left[ \sum_{u \in V(G)} d_u^2x_u^2 + d_1^2 \sum_{u \in V(G)} x_u^2 \sum_{v : vu \in E(G)} d_v \right] \]
\[ = 2 \left[ \sum_{u \in V(G)} d_u^2x_u^2 + d_1^2 \sum_{u \in V(G)} x_u^2 m_u d_u \right] \]

where \( m_u \) = average degree of the vertices adjacent to \( u \).

So,
\[ \mu_1 \leq \sqrt{2 \sum_{u \in V(G)} (d_u^2 + d_u^2m_u d_u) x_u^2}. \]

Hence
\[ \mu_1 \leq \max \left\{ \sqrt{2(d_u^2 + d_u^2m_u d_u)} : u \in V(G) \right\}. \]

§3. Lower Bound for Spectral Radius of Graphs

In this section we establish a lower bound for the spectral radius \( \mu_1 \) of \( G \).

**Lemma 3.1** ([7][8]) Let \( M \) be real symmetric matrix with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \). Given a partition \( \{1, 2, \ldots, n\} = \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_m \) with \( |\Delta_i| = n_i > 0 \), consider the corresponding blocking \( M = (M_{ij}) \), so that \( M_{ij} \) is an \( n_i \times n_j \) block. Let \( e_{ij} \) be the sum of the entries in \( M_{ij} \) and put \( B = (x_{ij}^2 m_{ij}) \) i.e., \( x_{ij}^2 \) is an average row sum in \( M_{ij} \). Let \( \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_m \) be the eigenvalues of \( B \). Then the inequalities
\[ \lambda_i \geq \gamma_1 \geq \lambda_{n-m+i} \quad (i = 1, 2, \ldots, m) \]
hold. Moreover, if for some integer \( k, 1 \leq k \leq m, \lambda_i = \gamma_i \) for \( i = 1, 2, \ldots, k \) and \( \lambda_{n-m+i} = \gamma_i \) for \( i = k + 1, k + 2, \ldots, m \), then all the blocks \( M_{ij} \) have constant row and column sums.
Let $G$ be a connected graph with $n$ vertices and $m$ edges. Let $V_1 = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V_2 = \{v_{n_1+1}, v_{n_1+2}, \ldots, v_n\}$ be two partitions of vertices of graph $G$. Let

\[
\begin{align*}
    r_1 &= \frac{1}{n_1} \sum_{i, j = 1 \atop i \neq j}^{n_1} \min(d(v_i), d(v_j)), \\
    r_2 &= \frac{1}{n-n_1} \sum_{i, j = 1 \atop i \neq j}^{n-n_1} \min(d(v_{n_1+i}), d(v_{n_1+j})), \\
    k_1 &= -\frac{1}{n_1} \sum_{i, j = 1 \atop i \neq j}^{n-n_1} \min(d(v_i), d(v_{n_1+j})), \\
    k_2 &= -\frac{1}{n-n_1} \sum_{j = 1 \atop j \neq \{i \atop i \in 1, 2, \ldots, n\} \atop i \neq j}^{n-n_1} \min(d(v_{n_1+i}), d(v_j)),
\end{align*}
\]

where $d(v)$ is the degree of the vertex $v$ of $G$. Now we prove the following theorem.

**Theorem 3.2** Let $G$ be a connected graph with $n$ vertices and $m$ edges, then

\[
\mu_1 \geq \frac{1}{2} \left\{ d_2 + d_1 - r_2 - r_1 + \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2} \right\}.
\]

**Proof** Rewrite $L(G)$ as

\[
L(G) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.
\]

For $1 \leq i, j \leq 2$, let $e_{ij}$ be the sum of the entries in $L_{ij}$ and put $B = (e_{ij}/n_i)$. Then

\[
B = \begin{pmatrix} d_1 - r_1 & k_1 \\ k_2 & d_2 - r_2 \end{pmatrix},
\]

and so

\[
|\lambda I - B| = \begin{vmatrix} \lambda - (d_1 - r_1) & -k_1 \\ -k_2 & \lambda - (d_2 - r_2) \end{vmatrix}.
\]

Therefore we have

\[
\lambda = \frac{1}{2} \left\{ d_2 + d_1 - r_2 - r_1 \pm \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2} \right\}.
\]

Thus by Lemma 3.1 we get

\[
\mu_1 \geq \frac{1}{2} \left\{ d_2 + d_1 - r_2 - r_1 + \sqrt{(d_2 - d_1 - r_2 + r_1)^2 - 4k_1k_2} \right\}.
\]

\[\square\]

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References


I want to bring out the secrets of nature and apply them for the happiness of man. I don’t know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor.
Author Information

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