A NOTE ON SOME RELIABILITY PROPERTIES OF EXTREME VALUE, GENERALIZED PARETO AND TRANSFORMED DISTRIBUTIONS

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Abstract

In this note, we study a few reliability properties of extreme value, generalized Pareto and some transformations of these distributions having positive support. These distributions are used in modelling series and parallel systems when the number of components is large.

Keywords: Reliability Properties, Extreme Value Distributions, Generalized Pareto Distributions.

Introduction

Suppose that $X_1, X_2, \ldots$, denote independent and identically distributed (iid) random variables (rvs) representing life times of components with common distribution function (df) $F$. Suppose further that these life times correspond to a series system and that the number of components tends to be very large. We know that $X_1 \wedge X_2 \wedge \ldots$ is the life time of such a system and one is often interested in

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knowing reliability properties of such systems when the number of components becomes larger and larger. Here \( \wedge \) denotes minimum and later \( \vee \) denotes maximum. The dual of this situation is when the system is parallel and in this set-up the system life time is given by \( X_1 \vee X_2 \vee \ldots \). We refer to Barlow and Proschan (1975) wherein Chapter 8 discusses approximate distributions for these life times under suitable normalizations.

In the next section, we study reliability properties of the extreme value distributions and some transformations of these defined on positive support. In the third and last section, we recall some definitions and look at some reliability properties for generalized Pareto distributions (gPds) and some transformed distributions of these having positive support.

**Reliability properties of extreme value distributions (EVDs) and their transformations**

If \( M_n = \max\{X_1, \ldots, X_n\} \) and \( m_n = \min\{X_1, \ldots, X_n\} \), then \( P(M_n \leq t) = F^n(t), t \in \mathbb{R} \), and \( P(m_n \leq t) = 1 - F^n(t), t \in \mathbb{R} \), where \( F \) is survival function. Note that \( M_n \) and \( m_n \) denote, respectively, the life times of parallel and series systems of \( n \) components whose life times are \( X_1, \ldots, X_n \).

If the number of components increases indefinitely, then it is easily seen that the limiting dfs of \( M_n \) and \( m_n \) are degenerate or do not exist. In order to get approximate limiting dfs for these two useful random variables, one can normalize these rvs. If there exist normalizing constants \( a_n > 0, b_n \in \mathbb{R} \) and \( c_n > 0, d_n \in \mathbb{R} \), and non-constant dfs \( G \) and \( L \) such that

\[
\lim_{n \to \infty} P \left( \frac{M_n - b_n}{a_n} \leq t \right) = \lim_{n \to \infty} F^n(a_nt + b_n) = G(t), \quad t \in \mathbb{R},
\]

and

\[
\lim_{n \to \infty} P \left( \frac{m_n - d_n}{c_n} \leq t \right) = \lim_{n \to \infty} 1 - (1 - F(c_n t + d_n))^n = L(t), \quad t \in \mathbb{R},
\]

then it is known that \( G \) and \( L \) have to be extreme value limit laws for the maximum and minimum respectively and these give approximations to the
normalized life times of parallel and series systems as the number of components increase indefinitely. It is well known that \( G \) has to be only one of three types of extreme value dfs, viz.,

- the Fréchet law, \( G_{1,\alpha}(x) = \begin{cases} 0, & x \leq 0, \\ \exp\{-x^{-\alpha}\}, & x > 0; \end{cases} \)
- the Weibull law, \( G_{2,\alpha}(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x < 0, \\ 1, & x \geq 0; \end{cases} \)
- the Gumbel law, \( G_{3}(x) = \exp\{-\exp(-x)\}, \ x \in \mathbb{R}, \)

\( \alpha > 0 \) being a parameter, with respective pdfs,

- the Fréchet density: \( g_{1,\alpha}(x) = \begin{cases} 0, & x \leq 0, \\ \alpha x^{-(\alpha+1)} e^{-x^{-\alpha}}, & 0 < x; \end{cases} \)
- the Weibull density: \( g_{2,\alpha}(x) = \begin{cases} \alpha|x|^{\alpha-1} e^{-|x|^\alpha}, & x < 0, \\ 0, & 0 \leq x; \end{cases} \)
- and the Gumbel density: \( g_{3}(x) = e^{-x} e^{-e^{-x}}, \ x \in \mathbb{R}. \)

Here, we say that two dfs \( F \) and \( G \) are of the same type if \( F(x) = G(Ax + B) \) for all \( x \), for constants \( A > 0 \) and \( B \in \mathbb{R} \). And, \( L \) has to be only one of the following three types of dfs:

- the negative Fréchet law, \( L_{1,\alpha}(x) = \begin{cases} 1 - \exp\{-(-x)^{-\alpha}\}, & x < 0, \\ 1, & x \geq 0; \end{cases} \)
- the negative Weibull law, \( L_{2,\alpha}(x) = \begin{cases} 0, & x < 0, \\ 1 - \exp\{-x^\alpha\}, & x \geq 0; \end{cases} \)
- the negative Gumbel law, \( L_{3}(x) = 1 - \exp(-e^{x}), \ x \in \mathbb{R}, \)

with respective pdfs,
the negative Fréchet density: \( l_{1,\alpha}(x) = \begin{cases} \alpha |x|^{-\alpha-1} \exp\{-|x|^{-\alpha}\}, & x < 0, \\ 0, & 0 < x; \end{cases} \)

the negative Weibull density: \( l_{2,\alpha}(x) = \begin{cases} 0, & x < 0, \\ \alpha x^{\alpha-1} \exp\{-x^\alpha\}, & 0 < x; \end{cases} \)

and the negative Gumbel density: \( l_{3}(x) = \exp(-e^{-x} + x), \quad x \in \mathbb{R}. \)

It is easy to see that results for minima can be obtained by transformation from corresponding results for maxima. So, except in a few cases, we do not state results for minimum.

The Gumbel law and its transformations

Since the support of the Gumbel law is the entire real line, we study the positive Gumbel law in this subsection. Let \( T \) denote the positive Gumbel rv, with df

\[
H_T(t) = \frac{P(0 < T \leq t)}{P(T > 0)} = \frac{G(t) - G(0)}{1 - G(0)} = \begin{cases} 0, & t < 0, \\ \frac{e^{-e^{-t}} - e^{-1}}{1 - e^{-1}}, & 0 \leq t. \end{cases}
\]

The survival function (sf) and the probability density function (pdf) of \( T \) respectively are

\[
\overline{H}_T(t) = 1 - H_T(t) = \begin{cases} 1, & 0 < t, \\ \frac{1 - e^{-t}}{1 - e^{-1}}, & 0 \leq t; \end{cases}
\]

and \( h_T(t) = \frac{d}{dt} H_T(t) = \frac{e^{-t} - e^{-1}}{1 - e^{-1}}, \quad 0 < t. \)

The failure rate or hazard rate (hr) \( \lambda_T(.) \), and cumulative hazard rate (chr) \( \Lambda_T(.) \) of \( T \) are, respectively,

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\[ \lambda_T(t) = \frac{h_T(t)}{H_T(t)} = \frac{e^{-t} - e^{-t}}{1 - e^{-t}}, \quad 0 < t; \]
\[ \Lambda_T(t) = \int_0^t \lambda_T(u) du = -\log H_T(t), \quad 0 < t. \]

The reversed hazard rate (rhr) \( r_T(.) \), and cumulative reversed hazard rate (crhr) \( R_T(.) \) of \( T \) are, respectively,

\[ r_T(t) = \frac{h_T(t)}{H_T(t)} = \frac{d}{dt} \log H(t) = \frac{e^{-t} - e^{-t}}{e^{-t} - e^{-1}}, \quad 0 < t; \]
\[ R_T(t) = \int_t^\infty r_T(u) du = -\log H_T(t), \quad 0 < t. \]

Now to verify if \( T \) is IFR (DFR), it is enough to show that the first derivative of the hazard rate is greater than (less than) zero. We have

\[ \frac{d}{dt} \lambda_T(t) = \frac{(e^{-t} - 1)e^{-t} - e^{-t}}{1 - e^{-t}} + \left( \frac{e^{-t} - e^{-t}}{1 - e^{-t}} \right)^2 > 0, \]

and hence conclude that the positive Gumbel law is IFR.

Block and Savits (1998) have shown with proof that every distribution that has a DFR (decreasing failure rate) function on its interval of support also has a DRHR (decreasing reversed hazard rate) function. In particular, the exponential distribution and the Weibull and gamma distributions with shape parameters less than 1 are all DRHR distributions. These authors have also shown that the IFR Weibull, gamma, and log-normal distributions are also DRHR distributions. We have similar results for positive Gumbel law

\[ \frac{d}{dt} r_T(t) = \frac{(e^{-t} - 1)(e^{-t} - e^{-t})}{e^{-t} - e^{-1}} - \left( \frac{e^{-t} - e^{-t}}{e^{-t} - e^{-1}} \right)^2 < 0, \quad t \geq 0, \]

which implies that the IFR positive Gumbel law is also DRHR.

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Block and Savits (1998) also show, by an example, that not all non-negative rvs have a DRHR distribution. Using the following relation between \( r_T(t) \) and \( \lambda_T(t) \),

\[
r_T(t) = \frac{h_T(t)}{H_T(t)} = \frac{\lambda_T(t)}{\left(e^{\int_0^t \lambda_T(u)du}\right) - 1}, \quad t \geq 0,
\]

we conclude that if \( h_T \) is decreasing, then \( r_T \) is also decreasing. See Finkelstein (2002) for more details.

The mean residual (remaining) lifetime (MRL) and mean reversed residual (inactivity) lifetime MRRL (MIT) functions of \( T \) are respectively given by

\[
m_T(t) = E(T - t | T > t) = \frac{1}{1 - e^{-t}} \int_t^\infty (1 - e^{-u})du, \quad t \geq 0, \quad \text{and}
\]

\[
M_T(t) = E(t - T | T \leq t) = \frac{1}{e^{-t} - e^{-t_0}} \int_0^t (e^{-u} - e^{-1})du, \quad t \geq 0.
\]

The MRL and \( r_T \), and MRRL and \( rhr \) satisfy the relations,

\[
\lambda_T(t) = \frac{m'_T(t) + 1}{m_T(t)}, \quad r_T(t) = \frac{1 - M'_T(t)}{M_T(t)}, \quad t \geq 0.
\]

**The Gumbel law for minima**

If \( X \) has the Gumbel law, then \( Y = -X \) has the Gumbel law associated with the minima of normalized iid rvs, with df

\[
G_Y(y) = P(Y \leq y) = 1 - P(X \leq -y) = 1 - e^{-e^y}, \quad y \in \mathbb{R}.
\]

We define the positive Gumbel minima rv \( S \) having df

\[
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\]
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\[ H_S(t) = P(S \leq t) = P(Y \leq t | Y > 0) = \begin{cases} 
0, & t < 0, \\
1 - \frac{e^{-e^t}}{e^{-t}}, & 0 \leq t.
\end{cases} \]

Its sf is \( H_S(t) = 1 - H_S(t) = \frac{e^{-e^t}}{e^{-t}}, \ t \geq 0, \) and its pdf is \( h_S(t) = \frac{d}{dt} H_S(t) = \frac{e^{-e^t} - e^{-t}}{e^{-1} - e^{-e^t}}, \ 0 \leq t. \) Its hr is given by \( \lambda_H_S(t) = \frac{h_S(t)}{H_S(t)} = e^t, \ 0 \leq t, \) which is increasing in \( t, \) showing that \( S \) is IFR. The chr of \( S \) is given by \( \Lambda_H_S(t) = \int_0^t \lambda_H_S(u) du = -\log H_S(t), \ 0 \leq t. \) The rhr of \( S \) is \( r_H_S(t) = \frac{d}{dt} H_S(t) - \frac{h_S(t)}{H_S(t)} = \frac{e^{t-e^t} - (e^{t-e^t})^2}{e^{-1} - e^{-e^t}}, \quad 0 \leq t, \) which is decreasing in \( t, \) since \( \frac{d}{dt} r_H_S(t) = \frac{e^{-e^t} - (e^{t-e^t})^2}{e^{-1} - e^{-e^t}} < 0, \quad 0 \leq t; \) showing that \( S \) is DRHR. The chrh of \( S \) is given by \( R_S(t) = \int_t^\infty r_S(u) du = -\log H_S(t), \quad 0 \leq t. \) Thus the positive Gumbel minima rv \( S \) is IFR and DRHR.

The MRL and MRRL (MIT) of \( S \) are in order, as follows

\[ m_{H_S}(t) = E(S - t | S > t) = \frac{1}{H_S(t)} \int_t^\infty H_S(u) du, \]

\[ = \frac{1}{e^{-e^t}} \int_t^\infty e^{-e^u} du = e^{-t}, \quad 0 \leq t, \]

which is a decreasing function in \( t. \) And the MRRL (MIT) of \( S \) is

\[ M_S(t) = E(t - S | S \leq t) = \frac{1}{e^{-1} - e^{-e^t}} \int_0^t (e^{-1} - e^{-e^u}) du, \quad t \geq 0. \]

We have following relations between MRL and fr and MRRL and rhr of the positive Gumbel minima distribution:

\[ \lambda_S(t) = \frac{m'_S(t)}{m_S(t)}, \quad r_S(t) = \frac{1 - M'_S(t)}{M_S(t)}, \quad t \geq 0. \]

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The Fréchet laws

The hr of the Fréchet law for maxima is given by

$$\lambda_G(t) = \frac{g_{1,\alpha}(t)}{G_{1,\alpha}(t)} = \frac{\alpha t^{-(\alpha+1)} e^{-t^{\alpha}}}{1 - e^{-t^{\alpha}}} = \frac{\alpha}{\alpha+1} \cdot \frac{1}{e^{t^{\alpha}} - 1}, \quad 0 \leq t, \quad 0 < \alpha,$$

and its rhr is given by

$$r_G(t) = \frac{g_{1,\alpha}(t)}{G_{1,\alpha}(t)} = \frac{\alpha}{\alpha+1}, \quad 0 \leq t, \quad 0 < \alpha.$$

It is trivial that the rhr, $r_G(t)$ is decreasing as in $t$ and hence the Fréchet law is DRHR. We shall now study the IFR / DFR behaviour of the Fréchet law. We have

$$\log\{f(t)\} = \log\{\alpha t^{-(\alpha+1)} e^{-t^{\alpha}}\} = \log\{\alpha\} - (\alpha + 1) \log\{t\} - t^{\alpha}, \quad t \geq 0, \quad \alpha > 0,$$

so that

$$\frac{d}{dt} \log\{f(t)\} = -\frac{\alpha + 1}{t} + \frac{\alpha}{\alpha+1}, \quad t \geq 0, \quad \alpha > 0,$$

and

$$\frac{d^2}{dt^2} \log\{f(t)\} = \frac{\alpha + 1}{t^2} - \frac{\alpha(\alpha + 1)t^\alpha}{(\alpha+1)^2} = \frac{\alpha + 1}{t^2} \left(1 - \frac{\alpha}{t^\alpha}\right), \quad t \geq 0, \quad \alpha > 0,$$

which is

$$\begin{align*}
&\text{if } t > \alpha^{\frac{1}{\alpha}}, \\
&\text{if } t < \alpha^{\frac{1}{\alpha}}
\end{align*}$$

$$\Rightarrow \begin{align*}
&\text{log } f \text{ is convex if } t > \alpha^{\frac{1}{\alpha}}, \\
&\text{log } f \text{ is concave if } t < \alpha^{\frac{1}{\alpha}}.
\end{align*}$$

$$\Rightarrow \begin{align*}
&F \text{ is DFR if } t > \alpha^{\frac{1}{\alpha}}, \\
&F \text{ is IFR if } t < \alpha^{\frac{1}{\alpha}}.
\end{align*}$$

using Lemma 5.9 in page 77 of Barlow and Proschan (1975) and the following lemma.

If $F$ is a df with pdf $f$ and if $\log\{f(t)\}$ is convex on $[0, \infty)$, then $F$ is DFR.

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If \( \log \{ f(t) \} \) is convex, then \(-\log \{ f(t) \} \) is concave, and hence \( \frac{1}{f(t)} \) is PF\(_2\). So, for \( 0 < t < s \), and \( 0 < x \), we have

\[
\left| \frac{\frac{1}{f(t)} - \frac{1}{f(s)}}{\frac{1}{f(t+x)} - \frac{1}{f(s+x)}} \right| \leq 0 \quad \Leftrightarrow \quad \frac{1}{f(t)f(s+x)} - \frac{1}{f(s)f(t+x)} \leq \frac{1}{f(t)f(s)}.
\]

\[
\Leftrightarrow \quad \frac{f(t+x)}{f(t)} \leq \frac{f(s+x)}{f(s)},
\]

\[
\Leftrightarrow \quad \int_0^\infty \frac{f(t+x)dx}{f(t)} \leq \int_0^\infty \frac{f(s+x)dx}{f(s)},
\]

\[
\Leftrightarrow \quad \frac{F(t)}{f(t)} \leq \frac{F(s)}{f(s)} \Leftrightarrow \lambda_F(t) \geq \lambda_F(s) \Leftrightarrow Fis \ DFR.
\]

And the rhr of the Fréchet law for maxima is given by

\[
\tau_F(t) = \frac{f_{1,\alpha}(t)}{F_{1,\alpha}(t)} = \frac{\alpha t^{-(\alpha+1)}e^{-t^\alpha}}{e^{-t^\alpha}} = \frac{\alpha}{\alpha+1}, \quad t \geq 0, \quad \alpha > 0,
\]

so that the Fréchet law for maxima is DRHR always as \( \tau_F(t) \) is decreasing in \( t \).

Note that the Fréchet minima distribution with df \( F(t) = 1 - e^{-t^\alpha}, \ t \geq 0, \ \alpha > 0, \) is nothing but the Weibull df and the reliability properties of this df are well known and we will not repeat these here.

**Reliability properties of generalized Pareto distributions (gPds) and their transformations**

The gPds are limit laws of linearly normalized conditional excesses over a high threshold as the threshold tends to infinity. These have been used to model exceedances over high thresholds and were first introduced and studied by Balkema and de Haan (1974). It is well known that the extreme value laws are related to the gPds by the relation \( W(x) = 1 + \ln G(x), \ x \in \mathbb{R} \), where \( G \) is an extreme value law and \( W \) is the corresponding gPd.

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Corresponding to the three types of extreme value laws, there are three possible types of gPds, namely,

the Pareto law: \( W_{1,\alpha}(x) = \begin{cases} 0, & x < 1, \\ 1 - x^{-\alpha}, & 1 \leq x; \end{cases} \)

the negative Beta (1,1) law: \( W_{2,\alpha}(x) = \begin{cases} 0, & x < -1, \\ 1 - (-x)^{\alpha}, & -1 \leq x < 0, \\ 1, & 0 \leq x; \end{cases} \)

and the standard exponential law: \( W_{3}(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-x}, & 0 \leq x; \end{cases} \)

\( \alpha > 0 \) being a parameter, with respective pdfs

the Pareto density: \( w_{1,\alpha}(x) = \begin{cases} 0, & x < 1, \\ \alpha x^{-\alpha-1}, & 1 \leq x; \end{cases} \)

the negative Beta (1,1) density: \( w_{2,\alpha}(x) = \begin{cases} 0, & x < -1 & 0 \leq x, \\ \alpha(-x)^{\alpha-1}, & -1 \leq x < 0; \end{cases} \)

and the standard exponential density: \( w_{3}(x) = \begin{cases} 0, & x < 0, \\ e^{-x}, & 0 \leq x. \end{cases} \)

One can define gPds, replacing the extreme value distributions by their counterparts, which are limit laws of normalized minima of iid rvs. For more details about extreme value dfs and gPds, we refer to Resnick (1987), Kotz and Nadarajah (2000), Coles (2001) and Castillo et al. (2004). We next study some reliability properties of gPds and some of their transformations.

**The gPds and some transformations**

Note that the reliability properties of the exponential law which is a gPd is well known. We look at reliability properties of gPd \( 1 + \log G \), when \( G \) is the Gumbel law for minima, namely, \( G(t) = 1 - e^{-e^t}, \quad t \geq 0 \). We have the df, sf and the pdf of this gPd respectively given by
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\[ F(t) = 1 + \log \left\{ 1 - e^{-e^t} \right\}, \quad t \in \{ x \in \mathbb{R} : 1 + \log G(t) > 0 \}, \]

\[ \bar{F}(t) = -\log \left\{ 1 - e^{-e^t} \right\}, \quad t \in \{ x \in \mathbb{R} : 1 + \log G(t) > 0 \}, \]

\[ f(t) = -\frac{d}{dt} F(t) = \frac{e^t - e^{e^t}}{1 - e^{-e^t}}, \quad t \in \{ x \in \mathbb{R} : 1 + \log G(t) > 0 \}. \]

The fr of this gPd is given by

\[ \lambda_F(t) = \frac{e^t - e^{e^t}}{1 - e^{-e^t} - \log \{1 - e^{-e^t}\}}, \quad \text{with} \]

\[ \frac{d}{dt} \log f(t) = t - e^t - \log \{1 - e^{-e^t}\}, \quad \text{and} \]

\[ \frac{d^2}{dt^2} \log f(t) = -e^t - \frac{(1 - e^t)e^t - e^{e^t}}{1 - e^{-e^t}} - \left( \frac{e^t - e^{e^t}}{1 - e^{-e^t}} \right)^2 < 0, \]

and hence \( \log f \) is concave, so that \( f \) is PF \(_2\) and \( F \) is IFR.

Also the rhr of \( F \) is

\[ r_F(t) = \frac{e^t - e^{e^t}}{1 - e^{-e^t} + \log \{1 - e^{-e^t}\}}, \quad \text{with} \]

\[ \frac{d}{dt} f(t) = \frac{(1 - e^t)e^t - e^{e^t}}{1 - e^{-e^t}} - \left( \frac{e^t - e^{e^t}}{1 - e^{-e^t}} \right)^2 < 0, \]

which implies that \( r_F(\cdot) \) is decreasing and hence that \( F \) is DRHR.

**On the gPd laws**

More generally, if \( F(t) = 1 + \log G(t) \) is a gPd, then its sf is \( \bar{F}(t) = -\log G(t) \),

and its pdf is \( f(t) = \frac{g(t)}{G(t)} \), so that its fr is given by

\[ \lambda_F(t) = \frac{g(t)}{G(t) - \log G(t)}, \]

\[ = r_G(t) \frac{1}{\log G(t)}. \]

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So, we can conclude that $F$ is IFR if $G$ is IRHR, and conversely, $G$ is DRHR if $F$ is DFR.

Similarly, the rhr of $F$ is given by

$$r_F(t) = \frac{g(t)}{G(t)} \frac{1}{1 + \log G(t)},$$

$$= r_G(t) \frac{1}{1 + \log G(t)},$$

so that $F$ is DRHR if $G$ DRHR and conversely, $G$ is IRHR if $F$ is IRHR.

The gPd law associated with positive Gumbel maxima distribution

Define $F(t) = 1 + \ln G(t)$, with $G(t) = \frac{e^{-e^{-t}} - e^{-1}}{1 - e^{-1}}$, $t \in \{x \in \mathbb{R} : 1 + \ln G(t) > 0\}$. Then its df, sf and pdf are respectively given by

$$F(t) = 1 + \ln \left\{ \frac{e^{-e^{-t}} - e^{-1}}{1 - e^{-1}} \right\},$$

$$F(t) = -\ln \left\{ \frac{e^{-e^{-t}} - e^{-1}}{1 - e^{-1}} \right\},$$

$$f(t) = \frac{e^{-e^{-t}} - e^{-t}}{e^{-e^{-t}} - e^{-1}}, \text{ with}$$

$$\ln f(t) = \ln \left\{ \frac{e^{-e^{-t}} - e^{-1}}{e^{-e^{-t}} - e^{-1}} \right\},$$

$$\frac{d}{dt} \ln f(t) = (e^{-t} - 1) - \frac{e^{-t} - e^{-t}}{e^{-e^{-t}} - e^{-1}},$$

$$\frac{d^2}{dt^2} \ln f(t) = -e^{-t} - \frac{(e^{-t} - 1)(e^{-t} - e^{-t})}{e^{-e^{-t}} - e^{-1}} - \left( \frac{e^{-t} - e^{-t}}{e^{-e^{-t}} - e^{-1}} \right)^2 < 0,$$

so that $\ln f$ is concave or $f$ is PF. Thus the fr given by

$$\lambda_F(t) = \frac{e^{-t} - e^{-t}}{e^{-e^{-t}} - e^{-1}} - \ln \left\{ \frac{e^{-e^{-t}} - e^{-1}}{1 - e^{-1}} \right\},$$

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is increasing and \( F \) is IFR.

The rhr given by \( r_F(t) = \frac{e^{-t} - e^{-t}}{e^{-t} - e^{-t}} \) is decreasing in \( t \) and hence \( F \) is DRHR as

\[
\frac{d}{dt} f(t) = \frac{(-1 + e^{-t})e^{-t} - e^{-t}}{e^{-t} - e^{-1}} - \left( \frac{e^{-t} - e^{-t}}{e^{-t} - e^{-1}} \right)^2 < 0.
\]

The gPd law associated with Gumbel minima distribution

Defining \( F(t) = 1 + \log G(t), t \in \{ x \in \mathbb{R} : 1 + \log G(x) > 0 \} \), with \( G(t) = \frac{e^{-1} - e^{-t}}{e^{-1}} \), we have the df, sf, and pdf of \( F \) respectively given by

\[
F(t) = 1 + \log \left\{ \frac{e^{-1} - e^{-t}}{e^{-1}} \right\}, \quad F(t) = -\log \left\{ \frac{e^{-1} - e^{-t}}{e^{-1}} \right\},
\]

\[
f(t) = \frac{e^{-t}}{e^{-1} - e^{-t}}, \text{ with }
\]

\[
\frac{d}{dt} \ln f(t) = 1 - e^{-t} - \frac{e^{t} - e^{-t}}{e^{-1} - e^{-t}}, \text{ and }
\]

\[
\frac{d^2}{dt^2} \ln f(t) = -e^{-t} + \frac{e^{t} - e^{-t}}{e^{-1} - e^{-t}} + \left( \frac{e^{t} - e^{-t}}{e^{-1} - e^{-t}} \right)^2 > 0, \text{ so that the fr }
\]

\[
\lambda_F(t) = \frac{e^{-t}}{e^{-1} - e^{-t}} \frac{1}{-\ln \left\{ \frac{e^{-1} - e^{-t}}{e^{-1}} \right\}},
\]

is decreasing in \( t \) using Lemma 5.9 in page 77 of Barlow and Proschan (1975) as \( \ln f \) is convex.

The rhr

\[
r_F(t) = \frac{e^{t} - e^{-t}}{-\ln \left\{ \frac{e^{-1} - e^{-t}}{e^{-1}} \right\}}, \text{ is decreasing as }
\]

\[
\frac{d}{dt} f(t) = \frac{1 - e^{-t}}{e^{-1} - e^{-t}} - \left( \frac{e^{t} - e^{-t}}{e^{-1} - e^{-t}} \right)^2 < 0,
\]

and hence \( F \) is DRHR.

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The Pareto gPd law

Defining \( F(t) = 1 + \ln G(t), \ t \in \{x \in \mathbb{R} : 1 + \ln G(x) > 0\} \), with \( G(t) = e^{-t^{-\alpha}}, \ t \geq 0, \alpha > 0 \) the df, sf, pdf, fr and rhr of the Pareto law \( F \) are respectively given by

\[
F(t) = 1 - t^{-\alpha}, \quad F'(t) = t^{-\alpha}, \\
f(t) = \alpha t^{-(\alpha+1)}, \quad \lambda_F(t) = \frac{\alpha}{t}, \\
r_F(t) = \frac{\alpha}{t(t^{\alpha}-1)}.
\]

Hence the Pareto law is DFR and DRHR.

Similar to the above discussions, it is possible to study reliability properties of gPd laws associated with other distributions.

References


Coles Stuart (2001). An Introduction to Statistical Modelling of Extreme Values, Springer-Verlag


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