ON RELIABILITY CONCEPTS FOR DISCRETE LIFETIME RANDOM VARIABLES

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Abstract

Reliability theory is a branch of probability theory which deals with life times of product and processes. We look at some reliability concepts for discrete lifetime random variables which are useful in studying reliability when the process or product works in cycles. We state and prove a few new results.

1. Introduction and definitions of some reliability concepts

Discrete random variables (rvs) appear naturally while studying reliability of products and / or processes which work in cycles / periods wherein the random phenomenon occurs as counts. For example, rotations of a motor, shifts in a factory, etc. In this small note, in the first section, we look at various definitions of reliability concepts given by several authors. The consequences of the definitions on several discrete distributions is reported in Section 2. Our study necessitates the need for unifying these definitions and clarifying contradictory and incomplete definitions. Some of these is reported elsewhere. It is a practice in reliability to state and prove results for the discrete case separately. In the third and last section, we list out a few definitions of

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stochastic orders among discrete rvs and prove some new results. The continuous counterparts of these results are reported elsewhere.

Let X be a random variable (rv) taking values 1, 2, 3, \ldots, with respective probabilities \( p_1, p_2, p_3, \ldots \). Sometimes, it is convenient to include 0 as a value for the rv X. Unless stated otherwise, except in the last section of the article, we do not include 0. Some of the definitions have to be modified if X takes the value 0.

1.1 Definitions of a few reliability concepts:

1. **Probability mass function:** Abbreviated as pmf, \( p_k = P(X = k) \) is the probability that \( X \) takes the value \( k, k = 1, 2, \ldots \) (Xie, 2002, Cyril and Olivier, 2003, Kemp, 2004.)

2. **Reliability function:** \( R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) \), \( k = 1, 2, 3, \ldots \) is the reliability function with \( R_0 = 1 \) (Xie, 2002, Cyril and Olivier, 2003.) Kemp (2004) defines this as survival function \( S_0 = 1 \) and \( S_k = P(X \geq k) = \sum_{j=k}^{\infty} P(X = j) = R_{k-1} \).

3. **Failure rate function:** \( \lambda_k = \frac{P(X=k)}{P(X \geq k)}, k \geq 1 \) (Xie, 2002, Cyril and Olivier, 2003, Kemp, 2004). Another definition of failure rate is given by \( r_k = \ln \frac{R_{k-1}}{R_k} \) (Xie, 2002).

4. **Hazard function:** \( h_k = \sum_{j=1}^{k} \lambda_j, k \geq 1 \) (Kemp, 2004).

5. **IFR / DFR:**

a) \( X \) is IFR / DFR if its failure rate is increasing/decreasing (Xie, 2002).

b) **Logconcave / Logconvex:** \( X \) is logconcave/logconvex if \( \{ \frac{p_{k+1}}{p_k}, k \geq 1 \} \) is decreasing/increasing. Accordingly, the failure rate \( \lambda_k \) is increasing (IFR) / decreasing (DFR) (Cyril and Olivier, 2003).

c) Let \( \eta(k) = \frac{p_k - p_{k+1}}{p_k} \), \( \Delta \eta(k) = \eta(k + 1) - \eta(k) \), \( k \geq 1 \) If \( \Delta \eta(k) > 0 \) then \( \lambda_k \) is nondecreasing (IFR), if \( \Delta \eta(k) < 0 \) then \( \lambda_k \) is nonincreasing (DFR), and if \( \Delta \eta(k) = 0 \) then \( \lambda_k \) is constant (Kemp, 2004). Note that this is equivalent to the definition in Cyril and Olivier (2003) because of the following. Since \( \Delta \eta(k) = \frac{p_{k+1}}{p_k} - \frac{p_{k+2}}{p_{k+1}} \), with

\[\begin{align*}
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\end{align*}\]
$\frac{p_{k+1}}{p_k} = g(k)$, we have $\Delta g(k) = g(k) - g(k + 1)$. If $\Delta g(k) > 0$, then $g(k) > g(k + 1)$ so that $g \downarrow$ and $g$ is logconcave. If $\Delta g(k) < 0$, then $g(k) < g(k + 1)$ so that $g \uparrow$ and $g$ is logconvex.

6. IFRA / DFRA:

   a) If $\sum_{i=1}^{j} \frac{r_i}{j} \leq (\geq) \sum_{i=1}^{k} \frac{r_i}{k}$, then $X$ is IFRA/DFRA (Cyril and Olivier, 2003). Equivalently, $X$ is IFRA/DFRA if $\sum_{i=1}^{j} \frac{r_i}{j} \leq (\geq) \sum_{i=1}^{k} \frac{r_i}{k}$, $\forall k \geq j \geq 1 \Leftrightarrow -\frac{\ln R_j}{j} \leq (\geq) -\frac{\ln R_k}{k}$ (Xie, 2002).

   b) $X$ is IFRA/DFRA according as its hazard function is increasing/decreasing on average to mean, that is, if $\frac{H_k}{k+1} > (<) \frac{H_{k-1}}{k}$, $k \geq 2$, (Kemp, 2004).

7. NBU / NWU:

   a) $X$ is NBU/NWU if $R_j R_k \geq (\leq) R_{j+k}$, $\forall j \geq 1, k \geq 1$
   \hspace{1cm} $\Leftrightarrow \sum_{i=1}^{j} r_i \geq (\leq) \sum_{i=k+1}^{k+j} r_i$ (Xie, 2002).

   b) $X$ is NBU/NWU if the conditional survival probability at time $x$ for an item that has survived till time $t$ is less/greater than the survival probability at time $x$ for a new item, that is, $\frac{S_{t+x}}{S_t} < (> ) S_x$, $x > 0$, $t > 0$ (Kemp, 2004).

8. NBUE / NWUE: $X$ is NBUE/NWUE according as $\sum_{j=1}^{\infty} \frac{S_{k+j}}{S_k} > (<) \sum_{j=0}^{\infty} S_j$ (Kemp, 2004).

2. Examples and Discussion

Here, we look at several examples of discrete rvs and classify them according to the definitions given in Section 1.

1. Discrete uniform:

   a) Pmf is $p_k = \frac{1}{n}$ for $k = 1, 2, 3, \ldots n$. 

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b) According to Xie (2002), Cyril and Olivier (2003), reliability function
\[ R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \frac{k}{n} \]
and according to Kemp(2004), survival function
\[ S_k = \sum_{j=k}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k-1} P(X = j) = \frac{n-k+1}{n} \quad \text{with } S_0 = 1. \]

c) According to Xie (2002), hazard rate is
\[ r_k = \ln \left( \frac{R_{k-1}}{R_k} \right) = \ln \left( 1 + \frac{1}{n-k} \right). \]
As \( k \uparrow, \ln \left( 1 + \frac{1}{n-k} \right) \uparrow \) and therefore \( X \) is IFR.
According to Cyril and Olivier (2003), we have
\[ \lambda_k = \frac{P(X=k)}{P(X>k)} = \frac{1}{n-k+1}, \]
and
\[ p_{k+1} = \begin{cases} 1 & \text{if } 1 \leq k \leq n - 1, \\ 0 & \text{if } k = n. \end{cases} \]
So, \( p_{k+1} \downarrow \) as \( k \uparrow \), so that \( \lambda_k \uparrow \).
Therefore \( X \) is IFR.

According to Kemp (2004), hazard rate \( \lambda_k = \frac{P(X=k)}{P(X>k)} = \frac{1}{n-k+1} \). We have
\[ \eta(k) = \frac{p_{k+1}}{p_k} = \begin{cases} 0 & \text{if } 1 \leq k \leq n - 1, \\ 1 & \text{if } k = n. \end{cases} \]
\[ \Delta \eta(k) = \eta(k+1) - \eta(k) = \begin{cases} 0 & \text{if } 1 \leq k \leq n - 2, \\ 1 & \text{if } k = n - 1 \\ \text{not defined} & \text{if } k = n \end{cases} \]
Therefore we cannot say anything about \( X \) being IFR.

d) According to Xie (2002),
\[ \ln \frac{\bar{R}_k}{\bar{R}_j} = \ln \left( \frac{n-k}{n} \right)^{\frac{1}{k}} - \ln \left( \frac{n-j}{n} \right)^{\frac{1}{j}} \]
and hence discrete uniform is IFRA. Also, according to Kemp (2004),
\[ H_k = (k+1), H_{k-1} = \frac{k}{n-k+1} - \sum_{j=1}^{k-1} \frac{1}{n-j+1} > 0 \]
Therefore, discrete uniform is IFRA.

e) According to Xie (2002), we have
\[ R_i R_k - R_{i+k} \leq \frac{\lambda_j \lambda_k}{n^2} \geq 0. \]
Therefore, discrete uniform is NBU. Also, according to Kemp (2004),
\[ S_{t+k} - S_t S_k = \frac{1}{n-k+1} < 0 \]
so that discrete uniform is NBU.

f) According to Kemp (2004), we have
\[ \sum_{i=0}^{\infty} S_{t+i} - S_t \sum_{j=0}^{\infty} S_j = \sum_{i=0}^{\infty} \frac{u-t-j+1}{n} - \frac{u-t+1}{n} \sum_{j=0}^{\infty} \frac{u-j+1}{n} < 0 \]
so that discrete uniform is NBUE.

2. Shifted geometric:

a) Pmf is \( p_k = p(1-p)^{k-1}, k = 1, 2, 3, \ldots \).

b) According to Xie(2002), Cyril and Olivier(2003), reliability function
\[ R_t = P(X > k) = \sum_{j=t+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{t} P(X = j) = 1 - \sum_{j=1}^{t} p(1-p)^{j-1} = q^t. \]

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According to Kemp (2004), survival function
\[ R_{i-1} = S_k = \sum_{j=k}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k-1} P(X = j) = 1 - \sum_{j=1}^{k-1} p(1-p)^{j-1} = q^{k-1} \]
and \( S_0 = 1 \).

c) According to Xie (2002), hazard rate is \( r_k = \ln \left( \frac{R_{k-1}}{R_k} \right) = -\ln(1 - p) \), so that as \( k \uparrow \), \( r_k \) remains constant and hence shifted geometric is both IFR and DFR. Also, according to Cyril and Olivier (2003), \( \lambda_k = \frac{P(X = k)}{P(X > k)} = p \) and \( \frac{P_{k+1}}{P_k} = 1 - p \), which is a constant and according to Kemp (2004), \( \lambda_k = \frac{P(X = k)}{P(X > k)} = p \), and \( \eta(k) = \frac{p_k - p_{k+1}}{p_k} = p \). \( \Delta \eta(k) = \eta(k+1) - \eta(k) = p - p = 0 \), so that shifted geometric is both IFR and DFR.

d) According to Xie (2002), we have \[ \ln R_k \frac{\ln R_{k-j}}{j} = \ln((1 - p)^{1/2}) = 0 \). So, shifted geometric is both IFRA and DFRA. But, according to Kemp (2004), \( kH_k - (k+1)H_{k-1} = k^2p - (k+1)(k-1)p = p > 0 \), and hence shifted geometric is IFRA.

e) According to Xie (2002), we have \( R_iR_k - R_{i+k} = (1-p)^j (1-p)^k = (1-p)^{k+j} = 0 \) and hence shifted geometric is both NBU and NWU, but according to Kemp (2004), \( S_{i+k} - S_i, S_k = -p(1-p)^{k+i+2} < 0 \) and therefore it is NBU.

f) According to Kemp (2004), we have \[ \sum_{j=0}^{\infty} S_{i+j} - S_i, \sum_{j=0}^{\infty} S_j = \sum_{j=0}^{n-1+j+1} - \frac{n-1+j+1}{n} \sum_{j=0}^{n-1+j+1} n-j+1 < 0 \), so that shifted geometric is NBUE.

3 Shifted Poisson:

a) \( P_k = \frac{e^{-\lambda} \lambda^k}{(k-1)!}, k = 1, 2, 3, \ldots, \lambda > 0. \)

b) According to Xie (2002), Cyril and Olivier (2003), reliability function
\[ R_k = P(X > k) = \sum_{j=i+1}^{\infty} P(X = j) = 1 - \sum_{j=0}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} \frac{e^{-\lambda} \lambda^j}{(j-1)!}, \]
and according to Kemp (2004), survival function

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According to Cyril and Olivier (2003), hazard rate is
\[ \lambda_k = \frac{P(X=k)}{P(X \geq k)} = \frac{e^{-\lambda k^{-1}}}{\sum_{j=k}^{\infty} \frac{e^{-\lambda j^{-1}}}{(j^{-1})!}} \]
We have \( p_{k+1} = \frac{\lambda}{k} \) and as \( k \uparrow, \frac{p_{k+1}}{p_k} \downarrow \), so that \( p_k \) is logconcave. Therefore, shifted Poisson is IFR. According to Kemp (2004), we have \( \lambda_k = \frac{P(X=k)}{P(X \geq k)} = \frac{e^{-\lambda k^{-1}}}{\sum_{j=k}^{\infty} \frac{e^{-\lambda j^{-1}}}{(j^{-1})!}} \) and \( \Delta \eta(k) = \eta(k+1) - \eta(k) = \frac{k-\lambda}{k(k+1)} > 0 \). Therefore, \( \lambda_k \) is nondecreasing and shifted Poisson is IFR.

According to Xie (2002), we have
\[ \frac{\ln R_k}{k} = \frac{\ln R_j}{j} = 1 \ln \sum_{i=k}^{\infty} \frac{e^{-\lambda i^{-1}}}{(i^{-1})!} - \frac{1}{j} \ln \left( \sum_{i=j}^{\infty} \frac{e^{-\lambda i^{-1}}}{(i^{-1})!} + \sum_{i=k}^{j-1} \frac{e^{-\lambda i^{-1}}}{(i^{-1})!} \right) \leq 0 \]
Therefore, shifted Poisson is IFRA.

**Shifted binomial:**

a) Pmf is
\[ p_k = \binom{n}{k-1} p^{k-1} q^{n-k+1}, \quad k = 1, 2, 3, \ldots, n + 1, 0 < p < 1, q = 1 - p. \]

b) According to Xie (2002) and Cyril and Olivier (2003), reliability function
\[ R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \sum_{j=1}^{k} \binom{n}{j-1} p^{j-1} q^{n-j+1}. \]
According to Kemp (2004), survival function
\[ R_{k-1} = S_k = P(X \geq k) = \sum_{j=k}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \sum_{j=1}^{k} \binom{n}{j-1} p^{j-1} q^{n-j+1} \]
and \( S_0 = 1. \)

c) According to Cyril and Olivier (2003), hazard rate is \( \lambda_k = \frac{P(X=k)}{P(X \geq k)} \) and we have \( \frac{p_{k+1}}{p_k} = \left( \frac{k-k+1}{k} \right) q \downarrow \) as \( k \uparrow \). Therefore, shifted binomial is IFR.
d) According to Kemp (2004), \( \lambda_k = \frac{p^k(X=k)}{P(X> k)} = \frac{\binom{n}{k-1} p^{k-1} q^{n-k+1}}{1 - q} \) and \( \eta(k) = \frac{p^k(X=k)}{p(X>k)} = \frac{q^{k-1} p^{k-1} + q^k p^k}{q^{k-1} p^{k-1} + q^k p^k} \).

\[ \Delta \eta(k) = \eta(k+1) - \eta(k) = \frac{q^{k+1} p^{k+1} - q^{k} p^k}{q^{k+1} p^{k+1} + q^{k+1} p^{k+1}} > 0. \] Therefore, \( \lambda_k \) is nondecreasing and hence shifted binomial is IFR.

e) According to Xie (2002), \( \frac{ln R_k}{k} - \frac{ln R_{k+1}}{k+1} = \frac{1}{n} \sum \binom{n}{i-1} n^{-1} q^{i-1} p^{-1} + \frac{1}{n} \sum \binom{n}{i} p^{i-1} q^{n-i+1} \leq \frac{1}{n} \). Therefore, shifted binomial is IFRA.

4. Shifted negative binomial:

a) Pmf is \( p_k = \left( \frac{k+r-2}{k-1} \right) p^r q^{k-1} \) for \( k = 1, 2, 3, \ldots, r > 0, 0 < p < 1. \)

b) According to Xie (2002) and Cyril and Olivier (2003), reliability function

\[ R_k = P(X > k) = \sum_{j=k+1}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k} P(X = j) = 1 - \sum_{j=1}^{k} \binom{j+r-2}{j-1} p^j q^{j-1}. \]

According to Kemp (2004), survival function

\[ R_{k+1} = S_k = P(X \geq k) = \sum_{j=k}^{\infty} P(X = j) = 1 - \sum_{j=1}^{k-1} P(X = j) = 1 - \sum_{j=1}^{k-1} \binom{j+r-2}{j-1} p^j q^{j-1} \]

and \( S_0 = 1. \)

c) According to Cyril and Olivier (2003), hazard rate is \( \lambda_k = \frac{P(X=k)}{P(X>k)} \) and we have \( \frac{p_{k+1}}{p_k} = \left( 1 + \frac{r-1}{k} \right) q \downarrow \) as \( k \uparrow \). Therefore, shifted negative binomial is IFR. According to Kemp (2004), we have \( \lambda_k = \frac{P(X=k)}{P(X>k)} = \left( \frac{k+r-2}{k-1} \right) p^r q^{k-1} \)

Also, \( \eta(k) = \frac{p_k - p_{k+1}}{p_k} = \frac{k+q-kq-rq}{k} \) and

\[ \Delta \eta(k) = \eta(k+1) - \eta(k) = \frac{q(r-1)}{k(k+1)} > 0. \] Therefore, \( \lambda_k \) is nondecreasing and shifted negative binomial is IFR.

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According to Xie (2002), \( \frac{\ln R_k}{k} - \frac{\ln R_k}{i} = \)
\[
\frac{1}{k} \ln \sum_{i=k+1}^{\infty} \left( \begin{array}{c} i + r - 2 \\ i - 1 \end{array} \right) p_i^{r-1} - \frac{1}{i} \ln \sum_{i=j}^{\infty} \left( \begin{array}{c} i + r - 2 \\ i - 1 \end{array} \right) p_i^{r-1} + \sum_{i=k+1}^{\infty} \left( \begin{array}{c} i + r - 2 \\ i - 1 \end{array} \right) p_i^{r-1} \leq 0
\]

Therefore, shifted negative binomial is IFRA.

### 2.1 Summary of examples:

As is evident from the table below, different definitions may lead to different conclusions and there is a need to unify and clarify these definitions.

<table>
<thead>
<tr>
<th>Discrete rv</th>
<th>Xie ('02)</th>
<th>Cyr. &amp; Ol. ('03)</th>
<th>Kemp ('04)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform</td>
<td>IFR, IFRA, NBU</td>
<td>IFR</td>
<td>IFRA, NBU, NBUE</td>
</tr>
<tr>
<td>Shifted geom.</td>
<td>IFR &amp; DFR, IFRA &amp;</td>
<td>IFR &amp; DFR,</td>
<td>IFR &amp; DFR,</td>
</tr>
<tr>
<td></td>
<td>DFRA, NBU &amp; NWU</td>
<td>-</td>
<td>IFRA, NBU, NBUE</td>
</tr>
<tr>
<td>Shifted Poisson</td>
<td>IFRA</td>
<td>IFR</td>
<td>IFR</td>
</tr>
<tr>
<td>Shifted bin.</td>
<td>IFRA</td>
<td>IFR</td>
<td>IFR</td>
</tr>
<tr>
<td>Shifted neg. bin.</td>
<td>IFRA</td>
<td>IFR</td>
<td>IFR</td>
</tr>
</tbody>
</table>

### 3. Stochastic Orderings and Some New Results

#### 3.1 Some definitions of stochastic orderings among discrete rvs:

In this section, we list out a few definitions of stochastic orderings among discrete distributions and prove some new results.

Let \( X \) and \( Y \) be rvs taking values \( 0, 1, 2, \ldots \) and with respective pmfs
\[
p_X(k) = P(X = k) \quad \text{and} \quad p_Y(k) = P(Y = k)
\]
and survival functions
\[
P_X(k) = P(X > k), \quad P_Y(k) = P(Y > k), \quad \text{with} \quad P_X(0) = P_Y(0) = 1.
\]
Note that
\[
1 = P_X(0) \geq P_X(1) \geq P_X(2) \geq \ldots \quad \text{and} \quad \sum_{k=0}^{\infty} P_X(k) = \mu_X, \quad \text{the mean of} \ X.
\]
These are true for the rv \( Y \) also.

- If \( P(X > u) \leq P(Y > u), u \in R \), then \( X \) is said to be smaller than \( Y \) in the usual stochastic order, \( X \leq_{st} Y \). (Shaked and Shanthikumar, 1994).
If \( E(\phi(X)) \leq E(\phi(Y)) \) for all convex functions \( \phi : R \rightarrow R \), provided the expectations exist, then \( X \) is said to be smaller than \( Y \) in the convex order, \( X \leq_{\text{cx}} Y \) (Shaked and Shanthikumar, 1994).

\( X \) is said to be discrete new better (worse) than used in Laplace ordering (discrete NBUL (NWUL)) (see Yue and Cao, 2001) if

\[
\sum_{k=0}^{\infty} P_X(k+i)z^k \leq (\geq) \sum_{k=0}^{\infty} P_X(k)z^k, \quad 0 \leq z \leq 1, \; i = 0, 1, \ldots
\]

If

\[
\sum_{k=0}^{\infty} P_X(k)z^k \leq \sum_{k=0}^{\infty} P_Y(k)z^k \quad \forall \; z, \quad 0 \leq z \leq 1,
\]

then the probability generating function (pgf) of \( X \) is smaller than that of \( Y \) and \( X \) is said to be smaller than \( Y \) in the pgf ordering, denoted by \( X \leq_{\text{pgf}} Y \) (Alzaid and Proschan, 1991).

**A new result:**

We prove some new results in the next theorem where \( X_Y = X - Y | X > Y \) is the random life at random time \( Y \) and its survival function is

\[
P(X_Y > k) = P(X - Y > k | X > Y) = \frac{P(X > k, X > Y)}{P(X > Y)} = \frac{\sum_{l=0}^{\infty} P(X > k+y, Y = l) P(Y = l)}{\sum_{l=0}^{\infty} P(X > l, Y = l) P(Y = l)} , \quad k \geq 0
\]

**Theorem:** For discrete rvs \( X, Y \) taking non-negative integer values,

\[
X \text{ is NBU } \implies (i) X_Y \leq_{\text{d}} X,
\]

\[
(\ii) X_Y \leq_{\text{v}} X,
\]

\[
(\iii) X_Y \leq_{\text{p}} X,
\]

\[
(\iv) X \text{ is NBUL.}
\]

**Proof:** Let \( X, Y \) be discrete rvs taking non-negative integer values, and \( X \) be NBU.
Proof of (i):

\[ X \text{ is NBU } \Rightarrow P(X > k + l) \leq P(X > k)P(X > l), k, l \geq 0, \]
\[ \Rightarrow \sum_{l=0}^{\infty} P(X > k + l, Y = l) \leq \sum_{l=1}^{\infty} P(X > k)P(X > l, Y = l), k \geq 0, \]
\[ \Rightarrow \sum_{l=0}^{\infty} P(X > k + l | Y = l)P(Y = l) \leq \]
\[ P(X > k) \sum_{l=0}^{\infty} P(X > l | Y = l)P(Y = l), k \geq 0, \]
\[ \Rightarrow \frac{\sum_{l=0}^{\infty} P(X > k + l | Y = l)P(Y = l)}{\sum_{l=0}^{\infty} P(X > l | Y = l)P(Y = l)} \leq P(X > k), k \geq 0, \]
\[ \Rightarrow P(X_Y > k) \leq P(X > k), k \geq 0, \]
\[ \Rightarrow X_Y \leq_{\text{st}} X. \]

Proof of (ii): Arguing as above, we have

\[ X \text{ is NBU } \Rightarrow P(X_Y > k) \leq P(X > k), k \geq 0, \]
\[ \Rightarrow \sum_{k=1}^{\infty} P(X_Y > k) \leq \sum_{k=1}^{\infty} P(X > k), i \geq 0, \]
\[ \Rightarrow E(X_Y) \leq_{<} E(X), \]
\[ \Rightarrow X_Y <_{<} X. \]

Proof of (iii): Arguing as in the proof of (i) above, we have

\[ X \text{ is NBU } \Rightarrow P(X_Y > k) \leq P(X > k), k \geq 0, \]
\[ \Rightarrow \sum_{k=0}^{\infty} P(X_Y > k)z^k \leq \sum_{k=0}^{\infty} P(X > k)z^k, 0 < z \leq 1, \]
\[ \Rightarrow X_Y \leq_{\mu} X. \]
Proof of (iv):

\( X \) is NBU \( \Rightarrow \) \( P(X > k + l) \leq P(X > k)P(X > l) \), \( k, l \geq 0 \).

\[ \Rightarrow \sum_{k=0}^{\infty} P(X > k + l)z^k \leq \]

\[ P(X > l) \sum_{k=0}^{\infty} P(X > k)z^k, \quad 0 < z \leq 1, l \geq 0, \]

\[ \Rightarrow \quad X \text{ is NBUL}. \]

References


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