On rotations in a pseudo-Euclidean space and proper Lorentz transformations

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It is shown that in a general pseudo-Euclidean space $E^n_\epsilon$, 2-flats (planes) passing through the origin of the coordinate system may be classified into six invariant types and explicit formulas for "planar rotations" in these flats are obtained. In the physically important case of the Minkowski World $E^4_\epsilon$, planar rotations are characterized as rotationlike, boostlike and singular transformations and an invariant classification of proper Lorentz transformations into these types is given. It is shown that a general nonsingular proper Lorentz transformation may be resolved as a commuting product of two transformations one of which is rotationlike and the other boostlike while a singular transformation may be written as a product of two rotationlike transformations, each with a rotation angle $\pi$. Such a rotationlike transformation with angle $\pi$ is called "exceptional" following Weyl's terminology for similar transformations of $SO(3)$. In all cases, explicit formulas for the angles and planes of rotations in terms of the elements of a given Lorentz matrix are obtained and the procedure yields in a natural manner an explicit formula for the image of $L$ in the $D^{10}(D^{0})$ representation of $SO(3,1)$ which in turn leads to two more classification schemes in terms of the character $\chi$ of $L$ in the $D^{10}(D^{0})$ and the $D^{0}(D^{0})$ representations.

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1. INTRODUCTION

We consider here a general pseudo-Euclidean space $E^n_\epsilon$ spanned by the orthonormal basis $I_k$ with

$$I_k = (0,0,\ldots,0,1,0,\ldots,0); \quad p > k > 1,$$

$$I_k = (0,0,\ldots,0,i,0,\ldots,0); \quad p + 1 < k < n,$$

where only the $k$th-component of $I_k$ is nonzero and is either 1 or $i$ as indicated and the tilde denotes matrix transposition. An arbitrary vector of $E^n_\epsilon$ is given by

$$X = \sum x_k I_k,$$

where $x_k$ are all real. The scalar product of any two vectors $X$ and $Y$ is defined as

$$XY = \bar{Y}X = \sum g_{ij}x_iy_j,$$

where the diagonal matrix

$$(g_{ij}) = (I, I) = \text{diag}(1,1,\ldots,1,-1,-1,\ldots,-1),$$

with $p$ "plus-ones" and $(n-p)$ "minus-ones," is the metric induced on $E^n_\epsilon$ by the scalar product given in Eq. (1.3). $X$ and $Y$ are said to be orthogonal if $XY = 0$. Following a familiar nomenclature used in relativity theory, we define a vector $X$ to be timelike, null or spacelike according as

$$X^2 = \bar{X}X \leq 0.$$  

A non-null vector $X$ is said to be a unit vector if $X^2 = \pm 1$. Evidently the signature of $g_{ij}$ is $p - (n-p) = (2p - n)$. Here we may observe that the space $E^n_{\epsilon - p}$ is completely equivalent to $E^n_{\epsilon}$ except that $E^n_{\epsilon - p}$ has a metric with the opposite signature $(n - 2p)$ and the vector nomenclatures "timelike" and "spacelike" get interchanged.

In what follows, we adopt the following notation. As seen above, boldface capitals such as $I$, $X$, $Y$, $P$, $Q$, etc., denote $n$-dimensional column-vectors of $E^n_\epsilon$, and $I_k$ etc., the corresponding row vectors. Ordinary lightface capitals such as $L$, $S$, $A$, $I$, $S^b$, $S^s$, etc., denote $n \times n$ matrices. $E$ is the $n \times n$ unit matrix. Elements of a matrix $A$ are denoted by $A_{ij}$. In particular, these symbols denote, respectively, 4-vectors and 4 $\times$ 4 matrices in the case of the Minkowski world $E^4_\epsilon$. Lower case boldface letters such as $e$, $h$, $m$, $n$, $a$, $b$ and the two boldface script letters $\mathfrak{e}$ and $\mathfrak{E}$ denote 3-vectors.

2. TWO-FLATS PASSING THROUGH THE ORIGIN AND THEIR CLASSIFICATION

Any two-dimensional subspace of $E^n_\epsilon$ defined by the parametric equations

$$\mathbf{R} = \eta \mathbf{P} + \mu \mathbf{Q} + \mathbf{C},$$

may be called a 2-flat of $E^n_\epsilon$. Here $\mathbf{R}$, as usual, is the radius vector of a general point on the 2-flat defined by the fixed, linearly independent, vectors $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{C}$ of $E^n_\epsilon$ and $(\eta, \mu)$ are two real parameters taken from the range $-\infty < \eta, \mu < \infty$. A 2-flat passing through the coordinate origin is given by Eq. (2.1) with $\mathbf{C} = 0$, and such a 2-flat is completely determined by the linearly independent vector pair $(\mathbf{P}, \mathbf{Q})$ which however is not unique. Any other vector pair $(\mathbf{X}, \mathbf{Y})$ related to $(\mathbf{P}, \mathbf{Q})$ by

$$\mathbf{X} = a\mathbf{P} + b\mathbf{Q}, \quad \mathbf{Y} = c\mathbf{P} + d\mathbf{Q}.$$
serves equally well to define a 2-flat passing through the origin. This situation permits us to choose \( (X,Y) \) to be an orthogonal pair of vectors. To see this, let us suppose that \( (P,Q) \) is nonorthogonal. Then Schmidt’s orthogonalization procedure adapted to \( E^n_\varepsilon \) yields the following prescriptions for orthogonalization:

(i) If \( P \) and \( Q \) are both null, then \( X = P + Q \) and \( Y = P - Q \) are orthogonal and as \( X' = - Y' = 2\hat{P}Q \neq 0 \), \( X \) is timelike if \( Y \) is spacelike and vice versa.

(ii) If at least one of the two vectors \( (P,Q) \), say, \( P \), is nonnull, then \( X = P + Q \) and \( Y = Q - (\hat{P}P/P^2)P \) are orthogonal. Further \( Y^2 = Q^2 - (\hat{P}Q/P^2)P \), and this shows that when \( Q \) is null, \( X \) is timelike if \( Y \) is spacelike and vice versa. However, when \( Q \) is not null, \( Y \) may be timelike, spacelike or null depending on \( (P,Q) \).

Thus, the generating vector pair \( (X,Y) \), of a 2-flat through the origin, can always be chosen to be an orthogonal pair and we shall assume that it is so in the rest of this paper. In terms of an orthogonal pair \( (X,Y) \), the equation to a 2-flat through the origin becomes

\[
R = \eta X + \mu Y, \quad \tilde{X}Y = 0. \tag{2.4}
\]

It is easy to see that a general \( E^n_\varepsilon \) admits orthogonal vector pairs of the following six types: spacelike-spacelike (ss), spacelike-null (sn), spacelike-timelike (st), null-null (nn) null-timelike (nt), and timelike-timelike (tt). For example in \( E^2_\varepsilon \), we have \([0,0,0,0], [0,0,0,1], [1,0,0,0], [0,0,1,0], [1,0,0,1], [0,1,0,0], [1,0,1,0], [0,1,1,0], [1,1,0,0], [0,1,0,1], [1,1,0,1], [0,1,1,1], [1,1,1,0], [1,1,1,1]\) which are orthogonal vector pairs belonging to the types ss, st, st, nn, sn, and st respectively. However, we may observe that all these six types of orthogonal vector pairs exist only when \( g_{\varepsilon} \) has at least two positive terms and two negative terms, i.e., when \( n \geq 2 \) and \( p > 3 \). In particular, in \( E^n_{\varepsilon} \), of which the Minkowski world \( E^4_{\varepsilon} \) is a special case, only the three types ss, sn and st, of orthogonal vector pairs, are possible. This result can be seen easily as follows: Since \( p = n - 1 \), there is only one timelike member in every orthonormal basis of the type described in Eq. (1.1) and it is evident that given any timelike vector \( T \), one can always choose an orthonormal basis in which the components of \( T \) are given by \( \hat{T} = (0,0,\ldots,0,1) \). This form of \( T \) immediately shows that orthogonal vector pairs of the type tt and nt are impossible in \( E^n_{\varepsilon} \). Further, if \( X \) and \( Y \) are a pair of orthogonal vectors of which, say, \( X \) is null, then in a suitable orthonormal basis in which \( X \) has components given by \( \tilde{X} = (0,0,\ldots,0,x,0) \) and \( Y \) has components given by \( \tilde{Y} = (0,0,\ldots,0,1,0) \), \( \tilde{X}Y = 0 \) implies \( y_{n-1} = y_n \) and hence \( YY = y_1^2 + y_2^2 + \ldots + y_n^2 < 0 \). Therefore \( Y \) can only be spacelike, or if it is null, it is a constant multiple of \( X \) and this proves that orthogonal vector pairs of the type nn (and nt) are impossible in \( E^n_{\varepsilon} \). Thus, we see that in \( E^n_{\varepsilon} \), only three types of orthogonal vector pairs namely ss, st and sn, are possible.

We now prove that a 2-flat defined by Eq. (2.4) admits only one type of an orthogonal vector pair \( (X,Y) \). Let \( (X',Y') \) be a new pair of orthogonal vectors in the 2-flat defined by Eq. (2.4). Then they are related to \( (X,Y) \) by

\[
X' = \eta X + \mu Y, \quad Y' = \eta Y + \mu X, \quad \eta \mu' - \eta' \mu 
eq 0. \tag{2.5}
\]

Evidently we have

\[
(X')^2 = \eta^2 X^2 + \mu^2 Y^2, \quad (Y')^2 = (\eta')^2 X^2 + (\mu')^2 Y^2, \tag{2.6}
\]

and

\[
\tilde{X}Y' = \eta \eta Y^2 + \mu \mu Y^2 = 0. \tag{2.7}
\]

From these formulas it is evident that when \( (X,Y) \) belongs to the types ss, nn or tt, \( (X',Y') \) is also of the types ss, nn or tt respectively. When \( (X,Y) \) is of the type st, in the nontrivial case in which \( \eta \mu, \eta', \mu' \) are all nonzero, Eqs. (2.6) and (2.7) yield

\[
(X')^2 = (\eta/\mu')(\eta' - \eta)\mu X^2, \quad (Y')^2 = - (\eta'/\mu)(\eta' - \eta)\mu Y^2, \tag{2.8}
\]

which clearly show that \( X' \) and \( Y' \) are both non-null. Further, from Eq. (2.7) we also have

\[
(\eta'/(\mu')) \neq (- Y'^2/X'^2) > 0, \tag{2.9}
\]

as \( (X,Y) \) is an st-pair of orthogonal vectors. Therefore \( (\eta/\mu) \) and \( (\eta'/\mu') \) are of the same sign and from Eq. (2.8) it now follows that \( (X',Y') \) is also an orthogonal vector pair of the type st. In the two trivial cases in which either \( \eta = \mu = 0 \) or \( \eta' = \mu' = 0 \), this result is evident. Lastly, when \( (X,Y) \) is of the types nt or sn, with, say, \( X \) as the null vector, we have from Eqs. (2.5)–(2.7),

\[
(X')^2 = \mu^2 Y^2, \quad (Y')^2 = (\mu')^2 X^2, \quad \mu \mu' = 0. \tag{2.10}
\]

Both the cases \( \mu = 0, \mu' \neq 0 \) and \( \mu \neq 0, \mu' = 0 \) evidently lead again to an orthogonal vector pair \( (X',Y') \) in which one of the vectors is null (and is a constant multiple of the original null vectors \( X \)) and the other non-null vector has the same norm as that of the original null vector \( Y \).

Thus we see that every 2-flat defined by Eq. (2.4) admits precisely one type of an orthogonal vector pair only and hence can be characterized by the type of orthogonal vector pair it admits. This leads to a classification of these 2-flats into six types, corresponding to the six types of orthogonal vector pairs discussed above. These 2-flats may thus be designated as ss-2-flats, st-2-flats, etc.

It is also interesting to note the relation between these 2-flats and the null-cone

\[
\tilde{R}R = 0, \tag{2.11}
\]

passing through the origin. Evidently, the points at which the 2-flat of Eq. (2.4) intersects this null-cone are given by

\[
R = \eta X + \mu Y, \tag{2.12}
\]

where the \( (\eta,\mu) \) satisfy

\[
\eta^2 X^2 + \mu^2 Y^2 = 0. \tag{2.13}
\]

Thus we observe that tt and ss 2-flats intersect the null-cone at only one point, namely the origin with \( \eta = \mu = 0 \); sn or nt 2-flats touch the null-cone along a line and with \( X \) as the null vector in the orthogonal pair \( (X,Y) \) we find the equation to this line of tangency to be

\[
\mu = 0. \tag{2.14}
\]

A st-2-flat cuts the null-cone along two lines given by

\[
\eta = \pm (Y^2/X^2)^{1/2} \mu, \tag{2.15}
\]

and and nn-2-flat lies entirely on the null-cone. These results may be compared with the corresponding results for the case.
of \( E_4 \) given in Synge.\(^1\)

### 3. Planar Rotations in \( E^n_2 \)

We know\(^2\) that the linear homogeneous transformations \( A = (A_{ij}) \) in \( E^n_2 \), which leave the quadratic form

\[
\bar{X}X = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2,
\]

invariant, form the pseudo-orthogonal group \( G^n_2 \). The transformation matrix \( A \) is evidently orthogonal, i.e.,

\[
\bar{A}A = E,
\]

where \( E \) is the unit \( n \times n \) matrix. Moreover, since the \( n \)-vectors \( A X \) and \( X \) must have the same structure, given in Eq. (1.2), the elements \( A_{ij} \) of \( A \) must satisfy certain reality conditions. On block-dividing \( A \) into the form

\[
A = \begin{pmatrix} B & iC \\ iD & F \end{pmatrix},
\]

these reality conditions imply that \( B, C, D, F \) are all real matrices of orders \( p \times p, p \times (n-p), (n-p) \times p \) and \((n-p) \times (n-p)\) respectively. The set of all such \( A \) with \( \det(A) = +1 \), forms a subgroup \( G^n_2 \) of \( G^n_2 \).

We now consider certain abelian subgroups of \( G^n_2 \) which may be interpreted as groups of planar rotations. Let \((X, Y)\) be a linearly independent orthogonal pair of vectors of \( E^n_2 \). Further, let us assume that \( X \) and \( Y \), whenever non-null, have been normalized to \( \pm 1 \), so that in general the norms \( X^2 \) and \( Y^2 \) have only the values \( \pm 1 \), depending upon the nature of \( X \) and \( Y \). Then the skew-symmetric matrix

\[
S = XY - YX; \quad \bar{X} = 0,
\]

is evidently nonzero, and the matrix

\[
A = \exp(S\theta),
\]

where \( \theta \) is a scalar parameter, satisfies the conditions given in Eqs. (3.2) and (3.3). Moreover

\[
\det(A) = \det[\exp(S\theta)] = +1,
\]

as the trace of \( S \) is zero. Thus, the set of all such \( S \) defined by Eqs. (3.4)-(3.6), forms an abelian subgroup of \( G^n_2 \). Obviously \( S \) is the corresponding infinitesimal transformation. The orthogonal vector pair \((X, Y)\) defining \( S \) also defines a 2-flat, of \( E^n_2 \), passing through the origin. If \( \eta X + \mu Y \) is an arbitrary vector of this 2-flat (plane), then

\[
S(\eta X + \mu Y) = \mu Y^2 X - \eta X^2 Y,
\]

is again a vector of the same plane. If \( Z \), on the other hand, is a vector orthogonal to this plane, then we have

\[
SZ = 0; \quad (\bar{X}Z = \bar{Y}Z = 0).
\]

Thus \( A = \exp(S\theta) \) transforms the plane [defined by \((X, Y)\)] into itself and leaves invariant any vector orthogonal to it. Moreover, since \( A \) preserves the norm of a vector and \( \det(A) = +1 \), it may be regarded as a planar rotation (2-flat rotation).

We now evaluate the planar rotation matrices for each of the six types of 2-flats discussed in Sec. 2. For this, we note that

\[
S^2 = (XY - YX)(XY - YX) = -YX^2 X - X^2 YX,
\]

\[
S^3 = -(XY - YX)(XY^2 X + X^2 YX) = -XY^2 S.
\]

Using these in Eq. (3.5), we obtain six particular forms of \( A = \exp(S\theta) \), corresponding to the six types of planes as follows.

In an \( nn \)-2-flat, with \( X^2 = Y^2 = 0 \), we have

\[
A = E + S\theta = E + (XY - YX)\theta.
\]

It is interesting to note that this transformation, though not an identity transformation, leaves all vectors in the \( nn \)-2-flat unaltered. In an \( nt \)-2-flat, with \( X^2 = 0 \) and \( Y^2 = -1 \), we have

\[
A = E + S\theta + S^2 \theta^2 = E + (XY - YX)\theta + (XY - YX)\theta^2.
\]

In an \( sn \)-2-flat, with \( X^2 = 0 \) and \( Y^2 = 1 \), we have

\[
A = E + S\theta + S^2 \theta^2 = E + (XY - YX)\theta + (XY - YX)\theta^2.
\]

In a \( ss \)-2-flat, with \( X^2 = X = 1 \), we have

\[
A = E + S\sin\theta + S^2 (1 - \cos\theta) = E + (XY - YX)\sin\theta + (XY - YX)\cos\theta - 1.
\]

In a \( st \)-2-flat, with \( X^2 = Y^2 = -1 \), we have

\[
A = E + S\sin\theta + S^2 (1 - \cos\theta) = E + (XY - YX)\sin\theta + (XY - YX)\cos\theta - 1.
\]

We may now define the angle of rotation \( \varphi \) as the angle between an arbitrary non-null vector lying in the plane and its image under the rotation. To determine the angle between two non-null vectors, we use the definition given in Petrov\(^3\) and set

\[
\cos\varphi = \pm \frac{\mathbf{P} \cdot \mathbf{Q}}{||\mathbf{P}|| ||\mathbf{Q}||^{1/2}},
\]

where \( \varphi \) is angle between the \( n \)-vectors \( P \) and \( Q \) and the plus sign is taken when both \( P^2 \) and \( Q^2 \) are positive and the minus sign when both of them are negative. Obviously, this definition breaks down when a null vector is involved. If we now set \( \mathbf{P} = \eta X + \mu Y \) and \( \mathbf{Q} = A \mathbf{P} \) in Eq. (3.17), we get

\[
\cos\varphi = \pm \frac{\mathbf{P} \cdot \mathbf{Q}}{||\mathbf{P}|| ||\mathbf{Q}||^{1/2}},
\]

where we must choose the plus sign if \( \mathbf{P} \) is spacelike and the minus sign if \( \mathbf{P} \) is timelike. An angle of rotation \( \varphi \) is evidently defined by this formula for planar rotations in all 2-flats except the \( nn \)-2-flat. Using formulas (3.12)-(3.16) we find that \( \varphi = 0 \) for all rotations in the \( sn \) and \( nt \)-2-flats, \( \varphi = \theta \) for rotations in the \( ss \) and \( tt \)-2-flats and \( \varphi = i\theta \), a pure imaginary angle, for rotations in the \( st \)-2-flats. Lastly we note that the formulas in Eqs. (3.11)-(3.16) yield the following known special cases.

(i) Let \( X = I_x \) and \( Y = I_y \) be the unit vectors defining the \( k - l \) coordinate 2-flat of \( E^n_4 \). Then we obtain from the formulas (3.14)-(3.16), the following nonzero components of

\[
A_{kk} = A_{ll} = \cos\varphi, \quad A_{kl} = -A_{lk} = \sin\varphi,
\]

all other \( A_{mn} = 1 \),

where \( \varphi = \theta, \varphi = -\theta, \varphi = i\theta \) according as the coordinate 2-flat considered is \( ss \), \( tt \) or \( st \). This special form justifies...
the terminology rotations for the planar transformation $A$.

(ii) The rotation matrix for a rotation through a (real)
angle $\theta$ about the axis $\hat{a} = (a_1, a_2, a_3)$ in the Euclidean space

$$A = (A_{ij}) = \begin{pmatrix}
    a_1^2(1 - \cos\theta) + \cos\theta & a_2a_3(1 - \cos\theta) + a_3\sin\theta & a_3a_2(1 - \cos\theta) - a_2\sin\theta \\
    a_2a_3(1 - \cos\theta) - a_3\sin\theta & a_2^2(1 - \cos\theta) + \cos\theta & a_3a_1(1 - \cos\theta) + a_1\sin\theta \\
    a_3a_2(1 - \cos\theta) + a_2\sin\theta & a_3a_1(1 - \cos\theta) - a_1\sin\theta & a_1^2(1 - \cos\theta) + \cos\theta
\end{pmatrix}.
$$

(3.20)

This gives the well-known relation

$$A_{11} + A_{22} + A_{33} = 1 + 2\cos\theta,$$

(3.21)

for the determination of the angle of rotation for a given $A$, and a somewhat more specific form

$$A_{ij} - A_{ji} = 2a_i\sin\theta, \quad (i,j,k = 1,2,3 \text{ cyclic}),$$

(3.22)

to the relations, as given in Hamermesh\textsuperscript{2} or Wigner,\textsuperscript{6} for the determination of the axis of rotation. Further, the form of $A$
given by Eq. (3.20) [and also Eq. (3.14) directly] shows that $A$
will be symmetric for $\theta = \pi$ and assumes the form

$$\begin{pmatrix}
    2a_1^2 - 1 & 2a_1a_2 & 2a_1a_3 \\
    2a_2a_1 & 2a_2^2 - 1 & 2a_2a_3 \\
    2a_3a_1 & 2a_3a_2 & 2a_3^2 - 1
\end{pmatrix}.
$$

(3.23)

Then Eq. (3.22) becomes a trivial identity leaving the axis $\hat{a}$
undetermined. One has now to solve for the $a_i$ from the
elements of some row or column of the matrix (3.23), as for example from

$$A_{11} = 2a_1^2 - 1, \quad A_{12} = 2a_1a_2, \quad A_{13} = 2a_1a_3.$$  

(3.24)

In this case, $\det(E + A) = 0$ and $A$ would be what Weyl\textsuperscript{6} calls an exceptional matrix since such matrices do not fit directly into Cayley's parametrization of the rotation group $SO(3)$ and some effort is necessary, as Weyl puts it, to "render these exceptions ineffective."

We note that in the exceptional case, $\hat{a}$ as given by Eq. (3.24) is determined only up to an ambiguity in sign and this reflects the fact that in the topological representation of the rotation group by a sphere of radius $\pi$, diametrically opposite points are to be identified. As an example of an exceptional rotation matrix, we have

$$A = \frac{1}{3} \begin{pmatrix}
    0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\
    -\frac{1}{\sqrt{6}} & -1 & \frac{1}{\sqrt{2}} \\
    -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -1
\end{pmatrix}.
$$

(3.25)

which evidently has $\theta = \pi$ and $\hat{a} = \pm (1/\sqrt{2}, 1/\sqrt{3}, 1/\sqrt{6})$.

(3.26)

4. PROPER LORENTZ TRANSFORMATIONS IN $E_3^2$.

As an interesting application of the results of the previous section we devote the rest of this paper to a complete discussion of the physically important case of proper Lorentz transformations in the Minkowski world $E_3^1$. We show that any general nonsingular (or non-null) Lorentz transformation may be factored into a commuting product of two

planar transformations one of which is equivalent, by a proper Lorentz transformation, to a pure rotation and the other to a pure boost. We accordingly call these factor transformations rotationlike and boostlike. We also note that this resolution is a Lorentz invariant one and is to be contrasted with the other known result (see for example Anderson\textsuperscript{3}) that a Lorentz transformation may be expressed as a (noncommuting) product of a pure-rotation and a pure-boost. A singular (or null) Lorentz transformation, on the other hand, is shown to be factorizable into a product of two rotationlike transformations, each with a rotation angle $\pi$. These factor transformations are symmetric in Minkowski coordinates with $x_i = ict$, and show certain special features like the symmetric rotations of Sec. 3 and are therefore termed exceptional following Weyl. These do not appear to have been noticed in the literature. Our analysis also yields the necessary and sufficient conditions for a Lorentz transformation to be planar and leads to an invariant classification of Lorentz transformations of all types. We note here that a different scheme of classification and a prescriptive procedure for determining the angles and planes of rotation based on the antisymmetric part of a Lorentz transformation has been given by Bazanski\textsuperscript{8} who also gives a formula which is essentially the same as our Eq. (4.39). Since, however, a pure boost is symmetric in real coordinates and an exceptional transformation in Minkowski coordinates, one or the other has its antisymmetric part identically zero and is naturally excluded in his classification. Our procedure, on the other hand, covers all cases and yields explicit formulas for the angles and planes of rotation in terms of the elements of a given Lorentz matrix $L$. Moreover, based as it is on group-theoretical considerations, as we shall see in the next section, our method yields an explicit formula for the three-dimensional complex orthogonal representation of the Lorentz group $SO(3,1)$ which in turn leads to two other classifications schemes in terms of the characters $\chi(L)$ in the $D^{10}$ and $D^{10}$ representations.

It may be observed that there is a close analogy between the methods adopted here and those of electromagnetic theory because the algebra of the infinitesimal Lorentz transformations is the same as that of the electromagnetic field tensor. In the case of the nonexceptional (null as well as non-null) transformations, the analogy is with the reduction of an electromagnetic field at an event to its canonical form whereas in the exceptional case it is with the problem of the extraction of an extremal root of the electromagnetic energy tensor as in the RMW theory.\textsuperscript{9}

A proper Lorentz transformation in $E_3^1$ is represented

$$E_3 = E_3^1,$$

in the more familiar form as given in Jeffreys and Jeffreys,\textsuperscript{4} follows from Eq. (3.14) on writing $X \times Y = \hat{a}$. and we get

$$X = E_3Y \hat{a}.$$
by the $4 \times 4$ matrix $L = (L_{ij})$, $i, j = 1, 2, 3, 4$, with $L_{\alpha \gamma}$ and $L_{\nu \xi}$ for $\alpha = 1, 2, 3$ purely imaginary while all other elements are real and

$$LL' = E, \quad \det L = +1,$$  \hfill (4.1)

where $E$ is the $4 \times 4$ unit matrix. Note that we are following the conventions described in Sec. 1 and for $E_4'$ this simply means that we are employing the conventional Minkowski coordinates with $x_4 = i\epsilon t$. Since the proper Lorentz group is a Lie group, every proper Lorentz transformation $L$ may be written as

$$L = \exp(I),$$  \hfill (4.2)

where the infinitesimal transformation $I$ has the form

$$I = \begin{pmatrix}
0 & h_3 & -h_2 & i\epsilon_1 \\
-h_3 & 0 & h_1 & i\epsilon_2 \\
h_2 & -h_1 & 0 & i\epsilon_3 \\
-i\epsilon_1 & -i\epsilon_2 & i\epsilon_3 & 0
\end{pmatrix}.$$  \hfill (4.3)

We observe that $I$ has precisely the same structure as the electromagnetic field tensor with 3-vector fields $e$ and $h$. In this case we may look upon the Lie parameters $e_1, e_2, e_3, h_1, h_2, h_3$ as the components of two 3-vectors $e$ and $h$ which we shall call the parameter vectors of $L$. The eigenvalues of $I$ are thus $\pm i\theta_1, \pm \theta_3$, where

$$\theta_1^2 = \frac{1}{2}(h_1^2 - e_2^2) + \frac{1}{2}(h_2^2 - e_1^2) + (h_3^2 - e_3^2)^{1/2},$$  \hfill (4.4)

$$\theta_3^2 = -\frac{1}{2}(h_1^2 - e_2^2) + \frac{1}{2}(h_2^2 - e_1^2) + (h_3^2 - e_3^2)^{1/2},$$  \hfill (4.5)

and those of $L$ are $\exp(\pm \theta_1), \exp(\pm \theta_3)$. A Lorentz transformation for which $e = h = 0$ is evidently trivial (the identity transformation). There are two types of nontrivial Lorentz transformations. If the two invariants $e$ and $h$ are both zero, then the eigenvalues of $I$ are all zero and hence those of $L$ are all equal to 1. Such a Lorentz transformation is called singular (null) by Synge and its $L$ corresponds to a null electromagnetic field. A Lorentz transformation is nonsingular (non-null) if at least one of $h_1^2 - e_2^2$ and $h_3^2 - e_3^2$ is nonzero. The “singularity” of $L$ arises here from the fact that det $I = 0$ although det $L$ itself is $+1$.

We now invoke a basic result of electromagnetic theory (see for example Synge or Landau and Lifshitz) that there exist Lorentz frames in which a non-null electromagnetic field has its electric and magnetic vectors parallel. Adapted to our case, this means that if at least one of $h_1^2 - e_2^2$ and $h_3^2 - e_3^2$ is different from zero, there exists a Lorentz transformation $T$, $TT = E$ such that

$$I' = TIT = \begin{pmatrix}
0 & \theta_1 & 0 & 0 \\
-\theta_1 & 0 & 0 & 0 \\
0 & 0 & i\epsilon_3 & 0 \\
0 & 0 & 0 & -i\epsilon_3
\end{pmatrix},$$  \hfill (4.6)

where the common direction of the parameter vectors $e'$ and $h'$ of $I'$ has been chosen to be the $z'$ axis (of the new frame) rather than the $x$ axis as chosen by Synge in his discussion of the “geometry” of the electromagnetic field. This shows, in complete analogy with the four-dimensional rotation matrix, that every nonsingular Lorentz transformation may be brought to the canonical block-diagonal form (4-screw)

$$L' = TL \tilde{T} = (\exp(I') \tilde{T} = \exp(TTT') = \exp(I'),$$

$$= \begin{pmatrix}
\cos \theta_1 & \sin \theta_1 & 0 & 0 \\
-\sin \theta_1 & \cos \theta_1 & 0 & 0 \\
0 & 0 & \cosh \theta_3 & i\sinh \theta_3 \\
0 & 0 & -i\sinh \theta_3 & \cosh \theta_3
\end{pmatrix},$$  \hfill (4.7)

by a proper Lorentz transformation. We have thus given a direct proof of Synge’s theorem that every nonsingular proper Lorentz transformation is equivalent to a 4-screw. Writing $L' = R'B' = B'R'$, where

$$R' = \begin{pmatrix}
\cos \theta_1 & \sin \theta_1 & 0 & 0 \\
-\sin \theta_1 & \cos \theta_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$  \hfill (4.8)

and

$$B' = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & \cosh \theta_3 & i\sinh \theta_3 \\
0 & 0 & -i\sinh \theta_3 & \cosh \theta_3
\end{pmatrix},$$  \hfill (4.9)

we see that $R'$ is a pure rotation and $B'$ is a pure boost. Evidently $R'$ is a planar Lorentz transformation in the $s_2$-flat defined by the 4-vectors

$$\vec{X} = (1,0,0,0), \quad \vec{Y} = (0,1,0,0),$$  \hfill (4.10)

and the corresponding angle of rotation is $\theta_1$. Similarly, $B'$ is a planar transformation in the $s_2$-flat defined by the 4-vectors

$$\vec{Z} = (0,0,1,0), \quad \vec{W} = (0,0,0,1),$$  \hfill (4.11)

and has the angle of rotation $i\theta_3$. We thus have, in terms of the matrices of the original basis,

$$L = RB = BR', \quad R = \tilde{T}R' T, \quad B = \tilde{T}B'T',$$  \hfill (4.12)

where $R$ is equivalent to a pure rotation and $B$ to a pure boost by the Lorentz transformation $T$ and have, respectively, the same invariant angles $\theta_1$ and $i\theta_3$. We say that $R$ is rotation-like and $B$ is boost-like. Moreover, $R'$ and $B'$ are planar transformations since the latter are obtained from the former by a mere change of basis.

We observe from Eq. (4.6) that in the canonical basis (primed letters denote quantities in the canonical basis),

$$I' = \theta_1 S'_1 + i\theta_3 S'_3,$$  \hfill (4.13)

where

$$S'_1 = X\vec{Y}' - Y\vec{X}' = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},$$  \hfill (4.14)

and

$$S'_3 = Z\vec{W}' - W\vec{Z}' = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & i & 0
\end{pmatrix},$$  \hfill (4.15)

with $(X', Y', Z', W')$ defined in Eqs. (4.10) and (4.11), are the infinitesimal transformations in the $X' - Y'$ and $Z' - W'$
planes, respectively. Thus we get

\[ I = \theta_S + \theta_S^\dagger ; \quad S_\tau = \hat{T}_\tau S \hat{T}_\tau^\dagger ; \quad S_\beta = \hat{T}_\beta S \hat{T}_\beta^\dagger, \tag{4.16} \]

for the infinitesimal transformations in the original basis.

Here the suffixes \( r \) and \( b \) in Eqs. (4.13)–(4.16) are mere labels indicating rotation and boost respectively and no summation is implied by these repeated indices. We follow the same convention throughout the paper. From Eq. (4.16) it follows that

\[ S_r = X\hat{Y} - Y\hat{X}, \quad \text{and} \quad S_\beta = Z\hat{W} - W\hat{Z}, \tag{4.17} \]

and hence it is sufficient to compute only three distinct secular equations.

We have from (4.18)

\[ (4.19) \]

and \( \xi \) is known to be \( \exp(\pm \theta_S^\dagger) \), or \( \exp(\pm \theta_b^\dagger) \), and the orthonormal tetrad \( (X,Y,Z,W) \) giving the two blades (planes) of the transformation. We now proceed to determine these quantities in terms of the elements, \( L_\nu \) of the given Lorentz transformation \( L \).

We know that the characteristic equation of any \( n \times n \) matrix \( A \) is

\[ \sum_{r=0}^n (-1)^r \lambda^{n-r} p_r = 0, \tag{4.20} \]

where \( p_r \) is the sum of all \( r \) th order principal minors in the determinant of \( A \). Thus, we have for \( L \),

\[ \lambda^4 - \chi \lambda^3 + \xi \lambda^2 - \chi \lambda + 1 = 0, \tag{4.19} \]

where \( \chi = L_{11} + L_{22} + L_{33} + L_{44} = \text{spur} L \), \( \xi \) is the sum of all principal minors of the second order. In Eq. (4.19) the coefficient of \( (-\lambda) \) is also \( \chi \) because each element of a proper orthogonal matrix \( A (AA^\dagger = E) \) is equal to its cofactor. More generally, any minor of an orthogonal matrix is equal to its algebraic complement by Jacobi's theorem and hence it is sufficient to compute only three distinct second order minors of \( L \) to obtain \( \xi \). Since the roots of Eq. (4.19) are already known to be \( \exp(\pm i\theta_r) \) and \( \exp(\pm i\theta_b) \), we have from Vieta's formulas (see for example, Kurosh)[14]

\[ 2\cos\theta_r + 2\cos\theta_b = \chi, \tag{4.21} \]

\[ 2 + 4\cos\theta_r \cos\theta_b = \xi, \tag{4.22} \]

and hence

\[ \cos\theta_b = \frac{1}{2} (\frac{\chi}{\cos\sigma} - \sigma), \tag{4.23} \]

\[ \cos\theta_r = \sqrt{\frac{\xi}{\sin\sigma} - \sigma}, \tag{4.24} \]

where

\[ \sigma = \sqrt{\chi^2 - 4\xi + 8}^{1/2}. \tag{4.25} \]

These equations determine \( \theta_r \) and \( \theta_b \) explicitly in terms of \( L_\nu \). Since \( \theta_r \) is actually a rotation angle [see Eq. (4.8)], we take it to lie between 0 and \( \pi \). Similarly we take \( \theta_b \) to be positive. From Eq. (4.22) we have

\[ 1 + \frac{1}{2} \xi = 2 + 2\cos\theta_r \cos\theta_b, \]

and hence it follows that

\[ (\chi - 1 - \frac{1}{2} \xi) = 2(\cos\theta_b - 1)(1 - \cos\theta_r) > 0, \]

where the equality holds if and only if at least one of \( \theta_r \) or \( \theta_b \) is zero. But when \( \theta_r \) or \( \theta_b \) is zero, the transformation is evidently planar and thus

\[ \chi = 1 + \frac{1}{2} \xi, \tag{4.26} \]

is the necessary and sufficient condition for a proper Lorentz transformation to be planar. Further, if \( \theta_r = 0 \) and \( \theta_b \neq 0 \), we have \( \chi > 4 \) while if \( \theta_r \neq 0 \) and \( \theta_b = 0 \), \( \chi < 4 \). For \( \theta_r = \theta_b = 0 \), we have \( \chi = 4 \). We thus have the following invariant classification of proper Lorentz transformations: A proper Lorentz transformation is nonplanar or planar according as \( \chi < 1 + \frac{1}{2} \xi \), and a planar transformation is rotation-like, singular, or boost-like according as \( \chi \leq 4 \). In particular, a rotation-like transformation \( L \) with \( \theta_r = \pi \) has some special features due to the fact that \( \text{det}(L + E) = 0 \) for it and we call such a transformation exceptional following Weyl's characterization of similar transformations of the rotation group SO(3). A nonplanar proper Lorentz transformation (with \( \chi > 1 + \frac{1}{2} \xi \)) can always be written as a commuting product of two planar transformations one of which is rotation-like and the other boost-like.

Next, we consider the problem of expressing the two planes associated with \( L \) in terms of its elements. Let \( S = U V - V U \) be the infinitesimal transformation in the 2-flat determined by the orthogonal pair \( (U,V) \). As before, let us assume that \( U \) and \( V \), whenever non-null, have been normalized to \( \pm 1 \). One of these vectors, say \( U \), must necessarily be spacelike since a null or timelike vector cannot be orthogonal to another null or timelike vector. We now show that in the plane defined by the orthogonal pair \( (U,V) \), there always exists another orthogonal pair \( (P,Q) \) in which the spacelike vector \( P \) has its temporal component equal to zero. Consider first, the case in which both \( U \) and \( V \) are spacelike. Then \( (P,Q) \) given by

\[ P = \U \cos \varphi - \V \sin \varphi, \]

\[ Q = \U \sin \varphi + \V \cos \varphi, \]

are again orthogonal and yield the same \( S \). If we now choose \( \varphi \) such that \( \tan \varphi = (u_4/v_4) \), then \( p_4 \) would be zero \( (u_4, v_4 \) are respectively the time-components of \( U, V \) and \( P \). Next, if \( V \) is timelike, we consider the vector pair \( (P,Q) \) given by

\[ P = \U \cos \varphi - \V \sin \varphi, \]

\[ Q = -\U \sin \varphi + \V \cos \varphi. \]

This pair \( (P,Q) \) is also orthogonal with \( P \) spacelike and \( Q \) timelike and yields the same \( S \). But \( p_4 \) would be zero only if we can choose a \( \varphi \) such that \( \tan \varphi = (u_4/v_4) \). Since, however, \( |\tan \varphi| < 1 \), this choice is possible only if \( |u_4/v_4| < 1 \). This is certainly true, because

\[ u_4^2 + v_4^2 = (u_4 v_1 + u_4 v_2 + u_4 v_3)^2 \]

\[ \leq (u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2) \]

\[ = (1 + u_4^2)(1 - v_4^2) \]

\[ = 1 - u_4^2 + v_4^2 + u_4^2 v_4^2, \]

so that we have \( u_4^2 - v_4^2 > 1 \). Finally if \( V \) is a null vector, we consider the new orthonormal pair \( (P,Q) \) defined by

\[ P = \U - k \V, \]

\[ Q = \V, \]

which again lead to the same \( S \). Evidently we may choose \( k \) such that \( p_4 = 0 \). Thus we may assume, without loss of generality, that every infinitesimal Lorentz transformation of
the form
\[ S = U\tilde{V} - \tilde{V}U, \]
(4.27)
is such that the spacelike vector \( U \) has \( u_4 = 0 \). Since \( U \) is a unit vector, it follows that
\[ \tilde{U} = (\tilde{u},0), \]
(4.28)
where \( \tilde{u} \) is a unit 3-vector. We have assumed that when \( V \) is a non-null vector, \( V\tilde{V} = \pm 1 \) and in the case of a null \( V \) it is convenient to assume that it is “normalized” in the sense
\[ v_1^2 + v_2^2 + v_3^2 = v_4^2 = 1, \quad \tilde{V}V = 0. \]
(4.29)
Now we proceed to determine such a pair \((U,V)\) corresponding to a given planar infinitesimal transformation \( S \) whose parameter vectors satisfy the conditions
\[ h\cdot e = 0, \quad h^2 - e^2 = 1, \quad -1, \text{ or } 0. \]
(4.30)
With \( \tilde{U} = (\tilde{u},0) \) as given by Eq. (4.28) and \( \tilde{V} = (v_4,\tilde{u}_4) \), Eqs. (4.27) and (4.3) give
\[ (\tilde{u}\times v) = h, \quad \tilde{u}_4 = e. \]
(4.31)
This shows that the given \( S \) must have \( h\cdot e = 0 \) in agreement with Eq. (4.30). Moreover, since \( U \) and \( V \) are orthogonal we have
\[ \tilde{u}\cdot v = 0 \]
(4.32)
and this, in conjunction with Eqs. (4.30) and (4.31), implies
\[ h\times e = v \quad \text{and} \quad \tilde{u}_4 = |e|. \]
(4.33)
We thus have, for the orthonormal pair \((U,V)\),
\[ \tilde{U} = (\tilde{u},0), \quad \tilde{V} = (h\times e,i/e). \]
(4.34)
Observe that
\[ |h\times e|^2 - |e|^2 = |h|^2|e|^2 - |e|^2 = h^2 - e^2, \]
so that \( V \) is spacelike or timelike according as \( h^2 - e^2 = \pm 1 \), respectively, and null when \( h^2 - e^2 = 0 \). In the latter case we may factor out the common magnitude \( \theta = |e| = |h| \) from \( S \) and write it as \( S = S_3 \), where the parameter vectors of \( S_3 \) are unit 3-vectors. We then have
\[ S_3 = U\tilde{V} - \tilde{V}U, \quad \tilde{U} = (\tilde{u},0), \quad \tilde{V} = (h\times e,i). \]
(4.35)
If the given \( S \) has \( e = 0 \), then Eq. (4.34) fails to give \((U,V)\). \( S \) is then characterized only by \( h \) with \( h^2 = 1 \) and we take \( u_4 = 0 \) in accordance with Eq. (4.31). The 3-vectors \( u, v \) and \( h \) would then form an orthonormal triplet and we choose \((U,V)\) as
\[ \tilde{U} = (h_1^2 + h_2^2)^{-1/2}[h_2,-h_1,0,0], \quad \tilde{V} = (h_1^2 + h_2^2)^{-1/2}[-h_1 h_3 h_4, h_2 h_3, h_1^2 + h_3^2, 0]. \]
(4.36)
We thus note that the Eqs. (4.34)–(4.36) give the “orthonormal” pair \((U,V)\) explicitly in terms of the parameter vectors of an infinitesimal transformation \( S \) which satisfies the conditions given in Eq. (4.30). The problem is completely solved on showing that we can always extract two planar transformations \( S_3 \) and \( S_3^* \) both of which satisfy the conditions given in Eq. (4.30), from a given Lorentz transformation \( L \). Then the above formulas for \((U,V)\) determine the two 2-flats spanning the blades of \( L \).

A. Nonexceptional, nonsingular transformations

It follows from Eqs. (4.14) and (4.15) that the parameter vectors of \( S_3 \) and \( S_3^* \) satisfy
\[ e\cdot h' = 0, \quad e\cdot h'' = 0, \quad h^2 - e^2 = e^2 - h^2 = 1. \]
We also observe that
\[ S_3^2 = -S_3^*, \quad S_3^* = S_3', \quad S_3 S_3^* = S_3^* S_3 = 0. \]
These relations, being invariant, are also true in the original basis and we have
\[ e\cdot h, \quad e\cdot h'' = 0, \quad h^2 - e^2 = e^2 - h^2 = 1, \]
(4.37a)
\[ S_3^2 = -S_3, \quad S_3 S_3^* = S_3^* S_3 = 0. \]
(4.37b)
Therefore
\[ L = \exp(I) = \exp(\theta, S_3 + \theta, S_3^*), \]
(4.38)
where
\[ R = \exp(\theta, S_3) = E + S_3 \sin \theta_3 + S_3^2 (1 - \cos \theta_3) \]
and
\[ B = \exp(\theta, S_3^*) = E + S_3^* \sin \theta_3^* + S_3^2 (1 - \cos \theta_3^*). \]
On multiplying these and using Eq. (4.37), we obtain
\[ L = E + S_3 \sin \theta_3 + S_3^* \sin \theta_3^* + S_3^2 (1 - \cos \theta_3^*) + S_3^2 (1 - \cos \theta_3), \]
(4.39)
which is a polynomial of the second degree in the planar infinitesimal transformations \( S_3 \) and \( S_3^* \). We now proceed to determine \( S_3 \) and \( S_3^* \) from the given \( L \), or what is the same thing, we obtain explicit expressions for the parameter vectors \( e_3, h_3, e_3^* \) and \( h_3^* \) in terms of the elements \( L_{ij} \) of \( L \). Formulas given in Eqs. (4.34)–(4.36) will then give us the planes of rotation.

Let
\[ L_A = \begin{pmatrix} 0 & H_3 & -H_2 & iB_1 \\ -H_3 & 0 & H_1 & iB_2 \\ H_2 & -H_1 & 0 & iB_3 \\ -iB_1 & -iB_2 & -iB_3 & 0 \end{pmatrix}, \]
(4.40)
where \( 2B_\alpha = L_{\alpha\beta} - L_{\beta\alpha} \), \( (\alpha, \beta, \gamma) = 1,2,3 \) cyclic, and \( 2B_\alpha = L_{\alpha\beta} - L_{\beta\alpha} \), \( (\alpha, \beta, \gamma) = 1,2,3 \), be the antisymmetic part of \( L \). Equating the antisymmetric parts on both sides of Eq. (4.39) we have
\[ S_3 \sin \theta_3 + S_3^* \sin \theta_3^* = L_A, \]
(4.41)
\[ e_3 \sin \theta_3 + e_3^* \sin \theta_3^* = S, \]
(4.42)
the relation \( S_3 S_3^* = S_3^* S_3 \) leads to the following relations between the parameter vectors:
\[ e_3 \times e_3^* = h_3 \times h_3^* = m, \]
(4.43)
\[ e_3 \times h_3 = e_3^* \times h_3^* = n. \]
(4.44)
Forming the several expressions for \( m^2, n^2 \) and \( m \cdot n \), we obtain
\[ m^2 = e_3^2 e_3^* - (e_3 \cdot e_3^*)^2 = -e_3 \cdot h_3 (e_3^* \cdot h_3^*), \]
(4.45)
\[ n^2 = (e_3 \cdot e_3^*)(h_3 \cdot h_3^*) = e_3^* h_3^* - (e_3 \cdot h_3)^2, \]
(4.46)
\[ m \cdot n = -e_3 \cdot e_3^* (e_3 \cdot h_3^* = e_3 \cdot e_3^* (e_3 \cdot h_3), \]
(4.47)
\[ e \cdot e_b = 0. \]  
\[ \text{Similarly we obtain} \]  
\[ h_r \cdot h_b = 0. \]  
Equations (4.49) and (4.46) now show that \( n = 0 \), i.e., \( e_r \) is parallel to \( h_b \) and \( e_b \) is parallel to \( h_r \). But the two expressions for \( m \) show that while \( e_r \) is parallel to \( h_b \), \( e_b \) is antiparallel to \( e_r \). Further, Eq. (4.45) yields \( e_r^2 e_b^2 = h_r^2 h_b^2 \) and on using Eq. (4.37) we get \( e_r^2 = h_r^2 \) and \( e_b^2 = h_b^2 \), leading to  
\[ e_b = h_r, \quad h_b = -e_r. \]  
These relations show that \( S_r \) and \( S_b \) are duals of each other. Substituting Eq. (4.51) in Eqs. (4.41) and (4.42) and solving, we finally obtain  
\[ h_r = (\sin^2 \theta_r - \sinh^2 \theta_b)^{-1}(\sinh \theta_r \sin \theta_b + \mathcal{K} \sin \theta_b), \]  
\[ e_r = (\sin^2 \theta_r + \sinh^2 \theta_b)^{-1}(\sin \theta_r - \mathcal{K} \sin \theta_b), \]  
\[ h_b = -h_r. \]  
Since \( \theta_r \) and \( \theta_b \) are known from Eqs. (4.23)-(4.25), these equations give the parameter vectors directly in terms of the elements \( L_r \) of \( L \). In view of the properties given in Eq. (4.37), the formula given in Eq. (4.34) [or (4.36) if \( e_r \) or \( e_b \) is zero] now determines the orthonormal pair \((X=U, Y=V)\) yielding \( S_r \) as  
\[ S_r = XY - YX. \]  
Similarly we obtain the orthonormal pair \((Z=W, W=W)\) yielding \( S_b \) as  
\[ S_b = ZW - WZ. \]  
It also follows that the Lorentz transformation \( T \), which sends the given \( L \) into the canonical block-diagonal form given in Eq. (4.7), has for its rows, the row-vectors \((X, Y, Z, \tilde{N}) = -\mathcal{N}\). 

Although \( \theta_r \) and \( \theta_b \) are given by Eqs. (4.23)-(4.25), we may obtain expressions directly for \( \sin \theta_r \) and \( \cos \theta_b \) from Eqs. (4.52) and (4.53). On using the relations \( e_r \cdot e_b = 0 \) and \( h_r^2 - e_r^2 = 1 \), we can solve for \( \sin \theta_r \) and \( \cos \theta_b \), and we obtain  
\[ \sin^2 \theta_r = \frac{1}{2}(\mathcal{K}^2 - \mathcal{S}^2) + \left[ \left( (\mathcal{H}^2 - \mathcal{S}^2)^2 + (\mathcal{H}^2 - \mathcal{E}^2)^2 \right)^{1/2}, \right. \]  
\[ \sin^2 \theta_b = -\frac{1}{2}(\mathcal{H}^2 - \mathcal{S}^2) + \left[ \left( (\mathcal{H}^2 - \mathcal{S}^2)^2 + (\mathcal{H}^2 - \mathcal{E}^2)^2 \right)^{1/2}, \right. \]  
We observe that the whole procedure breaks down in two particular cases. First, if the vectors \( \mathcal{H} \) and \( \mathcal{E} \) of \( \mathcal{L}_4 \) are such that \( \mathcal{K}^2 = 0 \) and \( \mathcal{H}^2 - \mathcal{E}^2 = 0 \), then \( \sin^2 \theta_r = \sinh^2 \theta_b = 0 \) and Eq. (4.52) and (4.53) become meaningless. We shall see that this corresponds to the singular Lorentz transformation for which \( \theta_r = \theta_b = 0 \). The second case is that of the exceptional Lorentz transformation for which \( \mathcal{L}_4 \) is identically zero and the procedure evidently breaks down. It will be seen that \( \theta_r = \pi \) and \( \theta_b = 0 \) in this case, and we shall consider each of these cases separately, later.

From Eqs. (4.54) and (4.55), we see that  
\[ \sin \theta_r = \frac{\mathcal{H}^2 - \mathcal{S}^2}{\mathcal{H}^2 + \mathcal{S}^2} \]  
and  
\[ \sin \theta_b = \frac{\mathcal{E}^2 - \mathcal{S}^2}{\mathcal{H}^2 + \mathcal{S}^2} \]  
If on the other hand, \( \mathcal{K}^2 = 0 \) and \( \mathcal{S}^2 > \mathcal{H}^2 \), we have a boostlike transformation with \( \theta_r = 0 \), \( \sin \theta_b = \frac{\mathcal{E}^2 - \mathcal{S}^2}{\mathcal{H}^2 + \mathcal{S}^2} \) and  
\[ \mathcal{H}_b = (\mathcal{H}^2 - \mathcal{S}^2)^{-1/2} = -h_r, \]  
\[ \mathcal{E}_b = (\mathcal{E}^2 - \mathcal{S}^2)^{-1/2} = -e_r. \]  
Equations (4.56) and (4.57) determine the blades in the former case wherein \( \theta_r = 0 \) in the st-blade. Similarly Eqs. (4.58) and (4.59) give the blades in the latter case with \( \theta_r = 0 \) in the ss-blade.

### B. Singular Lorentz Transformations

In this case, the parameter vectors of the infinitesimal transformation satisfy  
\[ h^2 - e^2 = 0, \quad h e = 0, \]  
so that we have \( \theta_r = \theta_b = 0 \) and each eigenvalue of \( L \) is \( \pm 1 \). The infinitesimal transformation [see Eq. (4.3)] now corresponds to a null electromagnetic field and all that one can do is to carry out a rotation \( T \) of the spatial axes such that the vector \( h \), say, is along the \( z \) axis and \( e \) is along the \( x \) axis. Denoting the common magnitude of \( e \) and \( h \) by \( \theta \), we have the canonical form  
\[ I' = T IT = \theta \begin{bmatrix} 0 & 1 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} = \theta S', \]  
where  
\[ S' = X' \tilde{N}' - N' \tilde{X}', \quad \tilde{X}' = (1,0,0,0), \quad \tilde{N}' = (0,1,0,i), \]  
with \( X' \) evidently spacelike and \( N' \) null. We also have  
\[ (S')^2 = 0, \]  
so that we have in the original basis [see Eq. (3.13)]  
\[ L = \exp(\theta S) = E + \theta S + \frac{1}{2} \theta^2 S^2, \]  
as a planar transformation in the \( x \)-\( z \) plane determined by  
\[ X = \tilde{T} X', N = \tilde{T} N' \]  
and  
\[ S = X \tilde{X} - N \tilde{N} = T S' T. \]  
Since the eigenvalues of \( L \) are all \( \pm 1 \), we also have  
\[ X = 1 + \frac{i}{2} \theta = 4 \]  
as expected. On equating \( S' \) to the antisymmetric parts \( L_4 \) of \( L \), we obtain, because of Eq. (4.64),  
\[ \theta e = e, \quad \theta h = h, \]  
where \( e \) and \( h \) are the parameter vectors of \( S \) and \( \mathcal{E} \) and \( \mathcal{H} \) are the vectors occurring in \( L_4 \). Since \( e \) and \( h \) satisfy Eq. (4.60), we must have  
\[ S^2 - H^2 = 0, \quad C H^2 = 0. \]  
On the other hand, since \( T \) is merely a spatial rotation of

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the form
\[ T = \begin{pmatrix} \mathcal{R} & 0 \\ 0 & 1 \end{pmatrix} \]  
(4.68)

where \( \mathcal{R} \) is a three-dimensional spatial rotation matrix, it transforms the 3-dimensional vector \( e \) occurring in \( e' = \mathcal{R} e \) of \( S' \). Since \( e' \) is a unit vector as is clear from Eq. (4.61), \( e = -\hat{e} \) and hence \( h = \hat{h} \) are unit vectors. Therefore we have
\[ \theta = |\mathcal{R}| = |\mathcal{M}|, \quad \hat{e} = \mathcal{R} \hat{e}, \quad \hat{h} = \mathcal{R} \hat{h}, \]  
(4.69)

and the spacelike-null vector pair \((X, N)\) determining the plane of the singular transformation is given by, in view of Eq. (4.35),
\[ \mathbf{X} = (\hat{\mathcal{R}} \cdot 0), \quad \mathbf{N} = (\hat{\mathcal{R}} \times \hat{\mathcal{R}} \cdot 1), \]  
(4.70)

expressed in terms of the elements of \( L \). It is clear that the Lorentz transformation (rotation) \( T \) sending \( I \) to the canonical form \( I' \) of Eq. (4.61) is
\[ T = \begin{bmatrix} \hat{\mathcal{R}}_1 & \hat{\mathcal{R}}_2 & \hat{\mathcal{R}}_3 & 0 \\ \hat{\mathcal{R}}_1 & \hat{\mathcal{R}}_2 & \hat{\mathcal{R}}_3 & 0 \\ \hat{\mathcal{R}}_1 & \hat{\mathcal{R}}_2 & \hat{\mathcal{R}}_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \]  
(4.71)

where \( \hat{\mathcal{R}}_a \) and \( \hat{\mathcal{R}}_a' \) are the components of the corresponding unit 3-vectors.

C. Exceptional Lorentz transformations

We have called a rotation-like planar transformation with \( \theta_a = \pi \) and \( \theta_b = 0 \), an exceptional Lorentz transformation, and observe that it is symmetric as Eq. (4.39) with \( \theta_a = \pi \) and \( \theta_b = 0 \) reduces to
\[ L = E + 2S^2, \]  
(4.72)

where \( S^2 \) is symmetric by virtue of the antisymmetry of \( S_a \). We note, however, that an exceptional transformation is symmetric only in Minkowski coordinates and in a real coordinate system it is represented by an asymmetric matrix. Conversely, we show that we must have \( \theta_a = \pi \) and \( \theta_b = 0 \) for a symmetric Lorentz transformation. Taking the symmetric \( L \) to be
\[ L = \begin{bmatrix} L_{11} & q_3 & q_2 & ip_1 \\ q_3 & L_{22} & q_2 & ip_2 \\ q_2 & q_3 & L_{33} & ip_3 \\ ip_1 & ip_2 & ip_3 & L_{44} \end{bmatrix}, \]  
(4.73)

we have
\[ \chi = L_{11} + L_{22} + L_{33} + L_{44}, \]
and
\[ \frac{1}{2} \xi = (L_{11}L_{44} + p_1^2) + (L_{22}L_{44} + p_2^2) + (L_{33}L_{44} + p_3^2) = L_{44}(\chi - L_{44}) + (L_{44} - 1) = \chi L_{44} - 1, \]
so that
\[ 1 + \frac{1}{2} \xi = \chi L_{44}. \]  
(4.74)

But since \( L \) is symmetric \( (L = L') \), we have \( L L = L^2 = E \) so that the eigenvalues of \( L \) are equal to \( \pm 1 \) only. Since det(\( L \)) = + 1, all the eigenvalues must either be equal to + 1, or while two of them are equal to + 1, the other two must be equal to \( -1 \). In the former case \( \chi = 4 \) and in the latter \( \chi = 0 \). If \( \chi = 4 \), then Eq. (4.74) shows that
\[ \xi = 8L_{44} - 2 \text{ and thus we have } \]  
(4.75)

\[ \sigma^2 = \chi^2 - 4\xi^2 = 8(1 - L_{44}). \]  
But \( L_{44} > 1 \) for a proper transformation and hence we must have \( L_{44} = 1 \) and \( \sigma = 0 \) as \( \sigma^2 \) can never be negative. The vanishing of \( \sigma \) implies that \( \theta_a = \theta_b = 0 \) [see Eqs. (4.23) and (4.24)]. Since \( L \) is orthogonal, \( L_{44} = 1 \) implies that \( p = 0 \) and \( L = \) a pure rotation with \( \theta_a = 0 \) which simply means that \( L \) is the trivial identity transformation. On the other hand, if \( X = 0 \), we have \( \xi = -2 \) and \( \sigma = 4 \) leading to \( \theta_a = \pi \) and \( \theta_b = 0 \). Thus we have proved that any symmetric Lorentz transformation (in Minkowski coordinates with \( x_4 = ict \)) must necessarily be exceptional.

We now proceed to determine the plane of rotation of an exceptional Lorentz transformation \( L \). Since \( L_{44} = 0 \) for a symmetric \( L \), Eqs. (4.52) and (4.53) evidently fail to determine the parameter vectors \( e_r \) and \( h_r \). The problem now is to determine an antisymmetric \( S \), satisfying Eq. (4.72) from a symmetric \( L \) with \( L^2 = E \) and it is exactly analogous to the problem of the extraction of an “extremal root” of the electromagnetic energy tensor.

If an orthonormal pair of vectors \((X, Y)\) generate the plane of the transformation, which is an ss-2-flat, then
\[ S_e = X Y - Y X, \]
and
\[ S_h = -X X - Y Y. \]  
(4.75)

Therefore, \( S_e^2 X = -X \) and \( S_h^2 Y = -Y \) and it follows from Eq. (4.72) that \( X \) and \( Y \) are also eigenvectors of \( L \) belonging to the eigenvalue \( -1 \). If the given \( L \) has an especially simple structure, then one may determine a pair of orthonormal eigenvectors of \( L \) belonging to the eigenvalue \( -1 \) without much effort. Although the problem may be regarded as solved in principle by this prescription, we adopt another procedure, in conformity with our purpose of determining these vectors explicitly in terms of the elements \( L_{ij} \) of \( L \).

With \( h_r \equiv h = (h_1, h_2, h_3) \) and \( e_r \equiv e = (e_1, e_2, e_3) \) as the parameter vectors of \( S \), and \( L = (L_{ij}) \), Eq. (4.72) implies the following relations in which the Greek suffixes take the run of values 1, 2, 3, and the sequence \( \alpha, \beta, \gamma \) is cyclic in 1, 2, 3,
\[ L_{aa} = 1 + 2(e_\alpha^2 + h_\alpha^2 - h^2), \]  
(4.76)
\[ L_{44} = 1 + 2e_\alpha, \]  
(4.77)
\[ q_r = L_{0a} = L_{pa} = 2(h_\alpha h_\beta + e_\alpha e_\beta), \quad \alpha \neq \beta, \]  
(4.78)
\[ p_\alpha = L_{4a} = 2(e \times h_\alpha). \]  
(4.79)

On using \( h^2 - e^2 = 1 \) and \( h \cdot e = 0 \), we obtain from Eqs. (4.76) and (4.77)
\[ e_\alpha^2 + h_\alpha^2 = \frac{1}{2}(L_{aa} + L_{44}), \]  
(4.80)
while from Eqs. (4.78) and (4.79), we obtain
\[ 4e_\alpha h_\alpha = -q_\beta p_\beta + q_\beta p_\beta. \]  
(4.81)

Equations (4.80) and (4.81) now yield
\[ 2e_\alpha = \pm (f_\alpha \pm g_\alpha), \]  
(4.82)
\[ 2h_\alpha = \pm (f_\alpha \mp g_\alpha), \]  
(4.83)

where

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\[ f_{a} = 2^{-1/2}(L_{44} + L_{aa} - q^{a}p_{b} + q_{b}p^{a})^{1/2}, \]
\[ g_{a} = 2^{-1/2}(L_{44} + L_{aa} + q^{a}p_{b} - q_{b}p^{a})^{1/2}, \]
giving \( e \) and \( h \) explicitly in terms of the elements of \( L \). Since we obtain the same \( L \) on replacing \( S \) by \(-S\), in Eq. (4.72), the vectors \( h \) and \( e \) are determined only up to an indeterminacy in sign and we may take \( 2h_{a} = f_{a} + g_{a} \), in Eq. (4.83) and the solution of our problem will then be the \( h_{a} \) and \( e_{a} \) that are also consistent with Eq. (4.81) and the relation \( h^{2} - e^{2} = 1 \). The formulas of Eq. (4.34) now give the orthonormal spacelike vectors \((X, Y)\) determining the plane of rotation.

Lastly, we prove a result concerning a resolution of singular transformations. We saw that any nonplanar, nonsingular transformation may be written as a commuting product of two planar transformations one of which is rotationlike while the other is boostlike. We shall now obtain a somewhat similar result for singular transformations, namely, that every singular transformation is the product of two exceptional transformations.

From Eq. (4.61), we have, for a singular \( L \),
\[
TLT = L' = \exp(iT) = \begin{bmatrix} 1 & \theta & 0 & i\theta \\ -\theta & 1 - i\theta^{2} & 0 & -i\theta^{2}/2 \\ -i\theta & -i\theta^{2}/2 & 0 & 1 + i\theta^{2} \\ \end{bmatrix}. \tag{4.86}
\]
But we may express \( L' \) as
\[
L' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \end{bmatrix} \times \begin{bmatrix} -1 & -\theta & 0 & -i\theta \\ -\theta & 1 - i\theta^{2} & 0 & -i\theta^{2}/2 \\ 0 & 0 & 1 & 0 \\ -i\theta & -i\theta^{2}/2 & 0 & 1 + i\theta^{2} \\ \end{bmatrix} = L \cdot L', \tag{4.87}
\]
and observe that both \( L' \) and \( L' \) are symmetric and therefore exceptional. We thus obtain the resolution
\[
L = L_{1}L_{2}, \tag{4.88}
\]
where \( L_{1} = \bar{T}L \); \( T \) and \( L_{2} = \bar{T}L \); \( T \) are evidently exceptional proving the result stated above. Moreover, since the \( T \) as given in Eq. (4.71) has a particularly simple structure, we may even give \( L_{1} \) and \( L_{2} \) explicitly. It is easy to check that
\[
(L_{1})_{ab} = (L_{1})_{aa} = \mu_{a}\mu_{b} - \mu_{a}\mu_{b} - \mu_{a}\mu_{b} \}
\]
\[
(L_{2})_{ab} = (L_{2})_{aa} = \mu_{a}\mu_{b} - \mu_{a}\mu_{b} + \mu_{a}\mu_{b}, \tag{4.89}
\]
where \( \mu_{a}, \mu_{b}, \mu_{a} \) are the components of the respective unit vectors. A somewhat similar result for singular transformations

### D. Examples

As an illustration of the foregoing discussion, we consider a few simple examples of Lorentz transformations.

**Example (i):** Consider first,
\[
L = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & \gamma & 0 & i(\gamma^{2} - 1)^{1/2} \\ 1 & 0 & 0 & 0 \\ 0 & -i(\gamma^{2} - 1)^{1/2} & 0 & \gamma \\ \end{bmatrix}, \tag{4.91}
\]
where \( \gamma > 1 \). Clearly \( \gamma = 2\gamma \) and \( \xi = 1 \) so that \( \chi = 1 + \xi \) and \( \sigma = 2\gamma \) and the transformation is nonplanar with \( \theta = \pi/2 \) and \( \cos\theta_{0} = \gamma \). Forming the antisymmetric part of \( L \), we get \( \hat{N} = (0,1,0) \), \( \hat{M} = (0,0,1) \) and hence from Eqs. (4.52) and (4.53), \( h_{a} = e_{b} = (0,1,0) \). The formulas of Eq. (4.34) then give
\[
\hat{X} = (1,0,0,0), \quad \hat{Y} = (0,0,1,0), \quad \hat{Z} = (1,0,0,0), \quad \hat{W} = (0,0,0,1),
\]
where the first two vectors determine the plane of \( \theta = \pi/2 \) and the last two determine the plane of \( \theta_{a} = \cosh^{-1}\gamma \), a fact which is at once evident from the structure of \( L \).

**Example (ii):** As a second example, consider
\[
L = \begin{bmatrix} 1 + (\gamma - 1)w_{a}^{2}/\gamma^{2} & (\gamma - 1)w_{a}w_{b}/\gamma^{2} & (\gamma - 1)w_{a}w_{b}/\gamma^{2} & i\gamma w_{a}/c \\ (\gamma - 1)w_{a}w_{b}/\gamma^{2} & 1 + (\gamma - 1)w_{a}^{2}/\gamma^{2} & (\gamma - 1)w_{a}w_{b}/\gamma^{2} & i\gamma w_{a}/c \\ (\gamma - 1)w_{a}w_{b}/\gamma^{2} & (\gamma - 1)w_{a}w_{b}/\gamma^{2} & 1 + (\gamma - 1)w_{a}^{2}/\gamma^{2} & i\gamma w_{a}/c \\ -i\gamma w_{a}/c & -i\gamma w_{a}/c & -i\gamma w_{a}/c & \gamma \\ \end{bmatrix}, \tag{4.92}
\]
where \( \gamma = 1 - v^{2}/c^{2} \) is the Lorentz factor, \( L \) representing a pure boost with velocity \( v = (v_{x},v_{y},v_{z}) \) (see Möller and Synge). Here \( \gamma = 2\gamma + 1 \) and \( \frac{1}{2} \leq \gamma \leq 2 \) showing, as expected, that the transformation is planar and it is boostlike since \( \gamma > 1 \). Moreover, \( \sigma^{2} = \gamma^{2} - 4 + 8 = 4(\gamma - 1) \) so that \( \sigma = 2(\gamma - 1) \). Hence \( \cos\theta = \frac{1}{2} \). \( \gamma \). \( \gamma \) and \( \cos\theta_{0} = \gamma \). Forming the antisymmetric part of \( L \), we get \( \gamma = 0, \hat{M} = \gamma w/c, \) which yield through Eqs. (4.52) and (4.53), \( e_{a} = \gamma \gamma w/c, h_{a} = 0 \). But \( e_{a} = 1 \) as \( v^{2} = c \gamma^{2} - 1 \gamma^{2} \) and hence the 4-vectors defining the plane of the boost [see Eq. (4.34)] are \( \hat{X} = (v_{x},v_{y},v_{z},0) \) and \( \hat{W} = (0,0,0,1) \).

**Example (iii):** Another interesting example is
\[
L = \begin{bmatrix} 0 & \gamma & 0 & i(\gamma^{2} - 1)1^{1/2} \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -i(\gamma^{2} - 1)1^{1/2} & 0 & \gamma \\ \end{bmatrix}, \tag{4.93}
\]
where $\gamma > 1$, and clearly the transformation is planar as $\chi = \gamma + 1$ and $\frac{1}{2} \xi = \gamma$. However it is rotationlike, singular or boostlike according as $\gamma \leq 3$. But $\sigma^2 = (\gamma - 3)^2$ so that we must take $\sigma = 3 - \gamma$ or $\sigma = 3$ according as $\gamma \leq 3$. When $\gamma < 3$, we have $\sigma = \gamma - 3$; $\cos \theta = \frac{1}{2}(\xi - 1)$ and $\theta_b = 0$, for the resulting rotationlike transformation. When $\gamma > 3$, we have $\sigma = 3 - \gamma$ and the resulting boostlike transformation has $\theta_r = 0$ and $\sin \theta_b = \frac{1}{2}(\xi - 1)$. When $\gamma = 3$, $\sigma = 0$, $\theta_r = \theta_b = 0$ and the transformation is evidently singular. The vectors of the skew-symmetric part of $L$ are $\mathcal{S} = \{\gamma^2 - 1\}^{1/2} \{1,1,0,0\}$ and $\mathcal{S}^2 = \{0,1,\gamma + 1\}$. Equations (4.34) and (4.56)-(4.59) now yield the vectors defining the planes of the transformations to be

$$\mathcal{U} = 2^{-1/2}[1,1,0,0],$$

and

$$\mathcal{V} = \begin{cases} \frac{1}{2}(\gamma + 1)^{1/2} \times \{0,0,0,0\}, & \gamma < 3, \\ 2^{-1/2} \times \{1,1,0,0\}, & \gamma = 3, \\ \frac{1}{2}(\gamma - 3)^{1/2} \times \{0,0,0,0\}, & \gamma > 3. \end{cases}$$

We observe that the $L$ in question is obtained by carrying out a rotation through $\pi/2$ in the $x$-$y$ plane followed by a boost along the $x$ axis with a velocity $v$ corresponding to $\gamma$ and it is interesting that the composite transformation could be any of the three types depending on $\gamma$ and stays rotationlike up to a velocity as high as $c(8/9)^{1/2}$.

**Example (iv):** We finally consider one simple example of an exceptional transformation. For the symmetric matrix

$$L = \begin{pmatrix} 1 - 2\gamma^2 & 0 & 0 & 2i\gamma(\gamma^2 - 1)^{1/2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2i\gamma(\gamma^2 - 1)^{1/2} & 0 & 0 & 2\gamma^2 - 1 \end{pmatrix},$$

we evidently have $\chi = 0$, $\frac{1}{2} \xi = -1$ so that $\chi = 1 + \frac{1}{2} \xi = 0$ and hence we have $\theta_r = \pi$ and $\theta_b = 0$ in this case and the orthonormal vectors defining the plane of the rotation are obtained most simply as the eigenvectors of $L$ belonging to the eigenvalue $-1$. It is easy to verify that

$$\mathcal{U} = \{0,0,0,-i\beta\}, \quad \mathcal{V} = \{0,1,0,0\},$$

are the orthonormal vectors belonging to the eigenvalue $-1$ of $L$ spanning the plane of rotation. We note that we would have obtained the same vectors by the second method wherein the parameter vectors which follow from Eqs. (4.82) and (4.83) are $h = (0,\gamma,0)$ and $e = (0,0,\gamma,0)$.

### 5. CLASSIFICATION OF LORENTZ TRANSFORMATIONS IN THE $D^{10}$ AND $D^0$ REPRESENTATIONS

We now give two other classification schemes of Lorentz transformations in terms of the characters $\chi_+(L)$ and $\chi_-(L)$ of $L$, where $\chi_+(L)$ is the character of the element $L$ of $SO(3,1)$ in its three dimensional complex orthogonal representation $D^{10}$ (or $D^0$) and $\chi_-(L)$ is its character in the two-dimensional complex unimodular representation $D^0$ (or $D^0$). We shall do this by explicitly constructing from a given $L$, a complex orthogonal matrix $A$ which follows as an immediate consequence of the relations in Eqs. (4.16), (4.41), and (4.42). It is of interest to observe in this connection that Landau and Lifshitz use make use of the idea that a Lorentz transformation may be regarded as a rotation through a complex angle in three-dimensional space and give the transformation that corresponds to a velocity along the $x$ axis. Our matrix $A$ is just the appropriate generalization of their transformation to an arbitrary Lorentz transformation. Writing $h_r = e_b = \alpha$, $e_r = -h_b = \beta$ and denoting $h + ie$ by $f$, it follows from Eqs. (4.3) and (4.16), that

$$f = h + ie = (\theta_r + i\theta_b)\alpha + i\beta.$$  \hspace{1cm} (5.1)

We claim that the skew-symmetric matrix

$$\mathcal{F} = \begin{pmatrix} 0 & f_3 & -f_2 \\ -f_3 & 0 & f_1 \\ f_2 & -f_1 & 0 \end{pmatrix}$$

is the infinitesimal transformation in the $D^{10}$ representation of $SO(3,1)$. To see this, we observe that, if $I_{\alpha\beta}, I_{\gamma\delta}$ (or $\alpha, \beta, \gamma, \delta$ cyclic) are the infinitesimal transformations of the self-representation of $SO(3,1)$ in the coordinate planes, $I_{\alpha\beta} = -I_{\beta\alpha}$ would be the corresponding ones for the rotation group $SO(3)$ which is a direct product of two three-dimensional rotation groups whose infinitesimal transformations are $J_{\alpha} = \frac{1}{2}I_{\alpha\beta}$ and $K_{\alpha} = \frac{1}{2}(I_{\alpha\beta} + I_{\beta\alpha})$ so that we have the well-known relations

$$D^{10}(I_{\alpha\beta}) = D^{10}(J_{\alpha}) \times D^{10}(K_{\alpha}), \quad D^{10}(-I_{\alpha\beta}) = D^{10}(J_{\alpha}) \times D^{10}(K_{\alpha}).$$

Noting that $D^{10}(J) = D^{10}(K) = 0$ and $D^{10}(1) = 1$, we get

$$D^{10}(I_{\alpha\beta}) = D^{10}(I_{\alpha\beta} - I_{\alpha\beta}), \quad D^{10}(J_{\alpha}) = 0$$

and taking $J_{\alpha}$ in the self-representation of the rotation group $SO(3)$ we see that Eq. (5.4) yields for the representation matrix of $I_{\alpha}$ exactly the same matrix $\mathcal{J}$ of Eq. (5.2). We thus have $D^{10}(I) = \mathcal{J}$, and obtain

$$A = D^{10}(L) = \exp \mathcal{F} = \exp(\mathcal{F}_{\alpha\beta}) = E_3 + \sin \theta S_3 + (1 - \cos \theta) S_3^3.$$  \hspace{1cm} (5.5)
TABLE I. Classification schemes of proper Lorentz transformations.

<table>
<thead>
<tr>
<th>Representation</th>
<th>General nonplanar (Screw-like)</th>
<th>Planar transformations</th>
<th>Singular (Null)</th>
<th>Rotationlike</th>
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<tbody>
<tr>
<td>(D^{1/2}SO(3,1))</td>
<td>(\xi + 1 &lt; \chi) (\chi_x) complex</td>
<td>(\xi + 1 = \chi &gt; 4) (\chi_x &gt; 3) or (\chi_x &lt; -1)</td>
<td>(\xi + 1 = \chi = 4) (\chi_x = 3)</td>
<td>(\xi + 1 = \chi &lt; 4) (\chi_x = -1) or (\chi_x &lt; -2)</td>
</tr>
<tr>
<td>(D^{SO(3,3)})</td>
<td>(\xi + 1 &lt; \chi) (\chi_x) complex</td>
<td>(\xi + 1 = \chi &gt; 4) (\chi_x &gt; 3) or (\chi_x &lt; -1)</td>
<td>(\xi + 1 = \chi = 4) (\chi_x = 3)</td>
<td>(\xi + 1 = \chi &lt; 4) (\chi_x = -1) or (\chi_x &lt; -2)</td>
</tr>
<tr>
<td>(D^{P}SL(2,C))</td>
<td>(\chi_x) complex</td>
<td>(\chi_c &gt; 2) or (\chi_c &lt; -2)</td>
<td>(\chi_c = \pm 2) or (\chi_c &lt; -2)</td>
<td>(-2 &lt; \chi_c &lt; 2) (\chi_x = 0) Exceptional</td>
</tr>
</tbody>
</table>

where \(\theta = \theta_x + i\theta_y\) and \(S_x\) is the three-dimensional skew-symmetric matrix constructed from \(a = \alpha + i\beta\). We notice that \(A\) of Eq. (5.5) has determinant + 1 and has exactly the same structure as the \(A\) of Eq. (3.20) with the only difference that \(a\) and \(\theta\) are here complex so that \(A\) is complex orthogonal and unimodular. Since \(a^2 - \beta^2 = 1\) and \(\alpha - \beta = 0\), \(a\) is a unit vector and is an eigenvector of \(A\) of Eq. (5.5) belonging to the eigenvalue + 1 and may thus be regarded as the complex axis of rotation. We have from Eqs. (4.41) and (4.42)

\[
\alpha = \alpha + \beta = (\sin \theta_x + \sinh \theta_y)^{-1} \left( A^2 - B^2 + 2i A\cdot B \right) - 1/2 (A^2 + B^2),
\]

and \(\theta = \cos^{-1}(\chi - \sigma)/4 + \cosh^{-1}(\chi - \sigma)/2\) from Eqs. (4.23) and (4.24) so that all the elements of \(A = D^{10}(L)\) are expressed explicit in terms of the given Lorentz transformation \(L\). We note that, since the proper Lorentz group \(SO(3,1)\) and the complex orthogonal group \(SO(3,3)\) are both six-parameter groups and \(I \rightarrow \mathbb{R}\) is a one-one mapping, there exists exactly one complex orthogonal unimodular \(A\) corresponding to any given \(L\), and conversely.

For a pure rotation, \(\theta_x = 0\), \(\beta = 0\), \(\sin \theta_y = |\mathbf{A}|\) so that we have \(\alpha = \mathbf{A}\) and we recover the formula (3.20). For a pure boost as given by Eq. (4.92), \(\theta = 0\), \(\mathbf{B} = 0\), \(\sinh \theta_y = |\mathbf{B}|\) and we have

\[
\mathbf{a} = \mathbf{B} \equiv (v_x, v_y, v_z) v^{-1},
\]
giving

\[
D^{10}(L) = \left[ \begin{array}{ccc}
1 + (\gamma - 1)(1 - v_x^2/v^2) & -v_x v_y/v^2(\gamma - 1) + i \gamma v_x/v^2 & -v_x v_z/v^2(\gamma - 1) + i \gamma v_z/v^2 \\
-v_x v_y/v^2(\gamma - 1) - i \gamma v_x/v^2 & 1 + (\gamma - 1)(1 - v_y^2/v^2) & -v_y v_z/v^2(\gamma - 1) + i \gamma v_z/v^2 \\
-v_x v_z/v^2(\gamma - 1) - i \gamma v_x/v^2 & -v_y v_z/v^2(\gamma - 1) - i \gamma v_z/v^2 & 1 + (\gamma - 1)(1 - v_z^2/v^2)
\end{array} \right].
\]

For a velocity along the \(x\) axis \(v_x = v, v_y = v_z = 0\) and Eq. (5.7) reduces to the transformation as given by Landau and Lifshitz.\(^{10}\)

If \(L\) is exceptional, we have \(A = B = 0\), \(\theta_x = 0\), \(\theta_y = \pi\) and as expected Eq. (5.6) fails to determine \(\alpha\). The vectors \(\alpha\) and \(\beta\) and hence \(\alpha\) must then be determined from Eqs. (4.82)-(4.85) and we have \(\pi a = \mathbf{f}\) and obtain with the corresponding \(S_x\),

\[
D^{10}(L) = E_3 + 2S^3_3, \tag{5.8}
\]

which follows from Eq. (5.5) with \(\theta = \pi\). We observe that it has the same structure as Eq. (3.23) but with complex \(\alpha\).

When \(L\) is singular, we have \(\theta_x = 0, |\mathbf{A}| = |\mathbf{B}|\) and \(\alpha = \mathbf{A} + i \mathbf{B}\) is now a null vector. With \(\theta = |\mathbf{A}| = |\mathbf{B}|\), we obtain with the appropriate \(S_x\),

\[
D^{10}(L) = E_3 + S_3 \theta + |S^3_3| \theta^2
\]

This completes the explicit construction of the matrices of the \(D^{10}\) representation of the proper Lorentz group \(SO(3,1)\). The complex conjugate \(D^{10}^\dagger\) representation is evidently realized by taking \(\theta = -\theta_x - i\theta_y\) and \(\alpha = \alpha - i\beta\). It is now easy to introduce a classification of proper Lorentz transformations based on the \(D^{10}\) or \(D^{10}^\dagger\) representations. We have seen already that any complex orthogonal unimodular \(A\) belonging to \(SO(3,3)\) corresponds exactly to one proper Lorentz transformation \(L\). Thus the trace \(\chi_x\) of \(A\) would be the character of \(L\) in the \(D^{10}(D^{10}^\dagger)\) representation of \(SO(3,1)\) and from the structure of the matrices \(A\) as given by the formulas (3.20) and (3.23) with complex elements and the formulas (5.7) and (5.9), we obtain immediately that the Lorentz transformation

\[
L \text{ that corresponds to } A \text{ is (i) a general nonplanar transformation if } \chi_x \text{ is complex or real and } \chi_x < -1, \text{ (ii) is planar if } \chi_x \text{ is real and } \chi_x > -1, \text{ and is rotationlike, singular, or boostlike according as } \chi_x \leq 3, \text{ and is exceptional if } \chi_x = -1.
\]

The classification according to the \(D^{P}\) or \(D^{Q}\) representations is also straightforward. We know that the two-dimensional complex unimodular group \(SL(2,C)\) which is the same as the \(D^{P}\) or \(D^{Q}\) representation provides a double-valued representation of \(SO(3,1)\) and we have by the Clebsch–Gordan theorem,

\[
D^{P} \times D^{Q} = D^{10} + D^{10}_0. \tag{5.10}
\]

If, therefore, \(\chi_x\) is the trace of any complex unimodular ma-
matrix \( C \) belonging to \( \text{SL}(2,\mathbb{C}) \), we have
\[
X_C = X_A + 1,
\]
and since there is exactly one \( L \) corresponding to \( \pm C \), we arrive at the classification: The Lorentz transformation \( L \) that corresponds to a given \( C \) is (i) a general nonplanar transformation if \( X_C \) is complex; (ii) is planar if \( X_C \) is real and is rotationlike, singular, or boostlike according as \( |X_C| \leq 1 \), and is exceptional if \( X_C = 0 \). We collect the three classification schemes in Table I.

Note added in proof: It is shown by Synge [J. L. Synge, Comm. Dublin Inst. for Adv. Stud. Ser. A 21, 22 (1972)] that with each proper Lorentz transformation, one can associate a pair of complex unit quaternions \( \pm q = \pm (a_0 + a_\mu e_\mu) \), where the \( e_\mu (\mu = 1,2,3) \) satisfy
\[
e_\mu^2 = -1, \quad e_\mu e_\nu = -e_\nu e_\mu = e_\rho (\mu, \nu, \rho = 1,2,3 \text{ cyclic}).
\]
Since the quaternion units \( e_\mu \) have an irreducible representation \( e_\mu \rightarrow i\sigma_\mu \) in terms of the Pauli matrices \( \sigma_\mu \), we obtain \( q \rightarrow a_\rho E - ia_\mu \sigma_\mu = C \), an element of \( \text{SL}(2,\mathbb{C}) \), and we recover the \( D^{10} \) representation of \( \text{SO} (3,1) \). Thus \( X_C = 2a_0 \) and we arrive at the classification: A proper Lorentz transformation \( L \) which corresponds to a given \( q \) is (i) a general nonplanar one if \( a_0 \) is complex (ii) is planar if \( a_0 \) is real and is boostlike, singular, or rotationlike according as \( |a_0| \geq 1 \) and is exceptional if \( a_0 = 0 \). Table I would therefore be augmented by the following row, with the same column headings

<table>
<thead>
<tr>
<th>Quaternion</th>
<th>( a_0 ) complex</th>
<th>( a_0 &gt; 1 ) or ( a_0 = -1 )</th>
<th>( -1 &lt; a_0 &lt; 1 )</th>
<th>( a_0 = 0 ) exceptional</th>
</tr>
</thead>
</table>

It is of interest to observe that the four classification schemes neatly taper off according to the characterizing integers 4,3,2,1, which are the dimensions of the corresponding representations on formally regarding the quaternion representation as one-dimensional.