On a Continued Fraction of Ramanujan*

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Abstract

In this paper, we establish an integral representation of a $q$-continued fraction of Ramanujan and obtain its some explicit evaluations. We also derive its relation with the Ramanujan-Göllnitz-Gordon continued fraction .

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1. Introduction

Srinivasa Ramanujan has made some significant contributions to the theory of continued fraction expansions. The most beautiful continued fraction

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expansions can be found in Chapters 12 and 16 of his second notebook [17]. The celebrated Rogers-Ramanujan continued fraction is defined by

\[ R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \cdots}}}, \quad |q| < 1. \] (1.1)

On page 46 in his ‘lost’ notebook [19], Ramanujan claims that

\[ R(q) = \frac{\sqrt{5} - 1}{2} \exp \left( \frac{-1}{5} \int_q^1 \frac{(1 - t)^5(1 - t^2)^5 \cdots dt}{(1 - t^5)(1 - t^{10}) \cdots} \right), \] (1.2)

where 0 < q < 1. (1.2) was proved by G. E. Andrews [6] and for other integral representations of theta-functions, see [3]. On page 365 of his ‘lost’ notebook, Ramanujan wrote five modular equations relating \( R(q) \) with \( R(-q), R(q^2), R(q^3), R(q^4) \) and \( R(q^5) \). Ramanujan eventually found several generalizations and ramifications of (1.1) which are recorded in his ‘lost’ notebook. These and related works may be found in the papers by S. Bhargava [9], S. Bhargava and C. Adiga [10], [11], R. Y. Denis [14], [15], [16].

On page 366 of his ‘lost’ notebook, Ramanujan investigated the continued fraction

\[ G(q) := \frac{q^{1/3}}{1 + \frac{q^2}{1 + \frac{q^4}{1 + \frac{q^6}{1 + \cdots}}}, \quad |q| < 1, \] (1.3)

which is known as Ramanujan’s cubic continued fraction. H. H. Chan [12] has established several modular equations relating \( G(q) \) with \( G(-q), G(q^2) \) and \( G(q^3) \). Chan and Sen-Shan Huang [13] studied the Ramanujan-Göllnitz-Gordon continued fraction

\[ H(q) := \frac{q^{1/2}}{1 + q + \frac{q^2}{1 + q^3 + \frac{q^4}{1 + q^5 + \frac{q^6}{1 + \cdots}}}, \quad |q| < 1. \] (1.4)

Ramanujan computed several numerical values of \( R(q) \). One such value is

\[ \frac{e^{-2\pi/5} \cdot e^{-2\pi} \cdot e^{-4\pi} \cdot e^{-6\pi}}{1 + \frac{1}{1 + 1 + 1 + \cdots}} = \frac{5 + \sqrt{5}}{2} - \frac{\sqrt{5} + 1}{2}. \]

Recently, Bruce C. Berndt and Chan [8], Chan [12], Chan and Huang [13], C. Adiga et al. [2], [4], [5] have established several new interesting evaluations of (1.1), (1.3) and (1.4).
Motivated by these works, in this paper, we study the Ramanujan continued fraction
\[
M(q) := \frac{q^{1/8}}{1 + \frac{-q}{1 + q} + \frac{-q^2}{1 + q^2} + \frac{-q^3}{1 + q^3} + \cdots}, \quad |q| < 1. \tag{1.5}
\]
In his first letter [18, p. xxviii] to Hardy, Ramanujan mentions the Roger-Ramanujan continued fraction identity and a few other identities. Further he claims that the Rogers-Ramanujan continued fraction is a particular case of
\[
\frac{1}{1 + \frac{ax}{1 + \frac{ax^2}{1 + \frac{ax^3}{1 + \cdots}}}},
\]
which itself is a particular case of
\[
\frac{1}{1 + \frac{ax}{1 + bx + \frac{ax^2}{1 + bx^2 + \frac{ax^3}{1 + bx^3 + \cdots}}}.
\] \tag{1.6}
Note that, putting \(a = -1, b = 1, x = q\) in (1.6) and then multiplying by \(q^{1/8}\) we obtain (1.5).

In Section 2, we obtain an integral representation for \(M(q)\). In Section 3, we derive a formula which helps us to obtain relations among \(M(q), M(q^2), M(q^n)\) and \(M(q^{2n})\). In Section 4, we establish some evaluations of \(M(q)\) and obtain a relationship between \(M(q)\) and \(H(q)\).

We close this introduction with some definitions which will be used in the sequel. In Chapter 16 of his second notebook [17], Ramanujan develops the theory of theta function and his theta function is defined by
\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.
\]
Let
\[
\varphi(q) := f(q, q) = \frac{(-q; -q)_\infty}{(q; -q)_\infty}, \tag{1.7}
\]
and
\[
\psi(q) := f(q, q^3) = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.8}
\]
where \((a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).\) The product representations of these theta functions can be derived by using the Jacobi triple product identity:
\[
f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \quad |ab| < 1. \tag{1.9}
\]
2. Integral Representation for $M(q)$

One of the fascinating continued fraction identity recorded by Ramanujan in his ‘lost’ notebook [19] is

$$
\frac{G(aq, \lambda q, b; q)}{G(a, \lambda, b; q)} = \frac{1}{1 + \frac{aq}{1 + \frac{bq}{1 + \frac{aq^2 + \lambda q^3}{1 + \frac{bq^2 + \lambda q^4}{1 + \cdots}}}}}
$$

(2.1)

where

$$
G(a, \lambda, b; q) = \sum_{n=0}^{\infty} \frac{q^n (-1)^n}{(q; q)_n (-bq; q)_n}
$$

and

$$(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \geq 1.
$$

For a proof of (2.1), see S. Bhargava and C. Adiga [10]. They have also proved that

$$
\frac{G(aq, \lambda q, b; q)}{G(a, \lambda, b; q)} = 1 + \frac{aq}{1 - \frac{aq}{1 - \frac{bq}{1 - \frac{aq}{1 - \frac{bq}{1 - \cdots}}}}}
$$

(2.2)

Letting $a$ to 0 and then setting $\lambda = -1, b = 1$ in (2.2), we deduce that

$$
\sum_{n=0}^{\infty} \frac{q^{n+1}(-1)^n}{(q^2; q^2)_n} = \frac{1}{1 + \frac{-q}{1 + \frac{-q^2}{1 + \frac{-q^3}{1 + \cdots}}}}
$$

(2.3)

Now, employing the $q$-binomial theorem

$$
\frac{(-b; q)_{\infty}}{(a; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{(-b/a; q)_n a^n}{(q; q)_n}
$$

in (2.3) and then multiplying both sides by $q^{1/8}$, we obtain a product representation of $M(q)$, namely,

$$
q^{1/8} \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{q^{1/8}}{1 + \frac{-q}{1 + \frac{-q^2}{1 + \cdots}}} = M(q).
$$

(2.4)

Suppose $0 < q < 1$. Using (2.4) we find that

$$
\log M(q) = \frac{1}{8} \log q + \sum_{n=1}^{\infty} \log(1 - q^{2n}) - \sum_{n=1}^{\infty} \log(1 - q^{2n-1}.
$$
Taking the derivative of both sides, we obtain that
\[
\frac{d}{dq} \log M(q) = \frac{1}{8q} - \sum_{n=1}^{\infty} \frac{2nq^{2n-1}}{1 - q^{2n}} + \sum_{n=1}^{\infty} \frac{(2n - 1)q^{2n-2}}{1 - q^{2n-1}}
\]
\[
= \frac{1}{8q} - \frac{1}{q} \sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1 - q^n}
\]
\[
= \frac{1}{8q} - \frac{1}{4q} \left[ \varphi^2(-q)\varphi^2(-q^3) - 1 - 12q^3 \frac{\psi'(q^3)}{\psi(q^3)} \right],
\]
where we have made use of Entry 3 (iv) in Chapter 19 [7, p. 223], and logarithmic derivative of (1.8). Integrating both sides of (2.5) and then exponentiating we obtain the following theorem:

Theorem 2.1. For \( 0 < q < 1 \),
\[
M(q) = \exp \int \left( \frac{1}{8q} - \frac{1}{4q} \left[ \varphi^2(-q)\varphi^2(-q^3) - 1 - 12q^3 \frac{\psi'(q^3)}{\psi(q^3)} \right] \right) dq,
\]
where \( \varphi(q) \) and \( \psi(q) \) are as defined in (1.7) and (1.8).

3. Relationships among \( M(q), M(q^2), M(q^n) \) and \( M(q^{2n}) \)

Ramanujan recorded many modular equations in [7, Chapters 18-21]. To briefly define a modular equation, we first write as usual
\[
(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)}.
\]
A modular equation of degree \( n \) is an equation relating \( \alpha \) and \( \beta \) which is induced by
\[
\frac{\binom{1}{r + \frac{1}{r}; 1; 1 - \alpha}}{\binom{1}{r + \frac{1}{r}; 1; 1}} = \frac{\binom{1}{r + \frac{1}{r}; 1; 1 - \beta}}{\binom{1}{r + \frac{1}{r}; 1; \beta}}
\]
where
\[
\binom{a}{b; c; x} = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k, \quad |x| < 1.
\]
Let $Z_1(r) = _2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \alpha\right)$ and $Z_n(r) = _2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; \beta\right)$ where $n$ is the degree of the modular equation. The multiplies $m(r)$ is defined by the equation

$$m(r) = \frac{Z_1(r)}{Z_n(r)}.$$ 

Now we prove a theorem which will be used to derive the relations among $M(q)$, $M(q^2)$, $M(q^n)$ and $M(q^{2n})$.

**Theorem 3.1.** If

$$q = \exp\left(-\pi \frac{_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha\right)}{_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right),$$

then

$$16 \frac{M^8(q^2)}{M^8(q)} = \alpha. \quad (3.1)$$

**Proof:** By (1.7) and (2.4),

$$\varphi(q)M(q^2) = q^{1/4} \frac{(-q; -q)_\infty (q^4; q^4)_\infty}{(q; -q)_\infty (q^2; q^4)_\infty} = q^{1/4} \frac{(-q^2; -q^2)_\infty (q^2; q^2)_\infty (-q^2; q^2)_\infty}{(q; q^2)_\infty (q; q^2)_\infty (-q^2; q^2)_\infty} = M^2(q). \quad (3.3)$$

We recall from [1, p. 36, Entry 25 (vii)] that

$$\varphi^4(q) - \varphi^4(-q) = 16q\varphi^4(q^2) = 16M^4(q^2). \quad (3.4)$$

Using (3.3) in (3.4), we find that

$$16M^4(q^2) = \frac{M^8(q)}{M^4(q^2)} \left(1 - \frac{\varphi^4(-q)}{\varphi^4(q)}\right). \quad (3.5)$$

From [7, Entry 5, p. 100], we know that the identity (3.1) implies that

$$\alpha = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \quad (3.6)$$
The claim now follows from (3.5) and (3.6). Let $\alpha$ and $q$ be related by (3.1). If $\beta$ has degree $n$ over $\alpha$, then from Theorem 3.1, we obtain that

$$16 \frac{M^8(q^{2n})}{M^8(q^n)} = \beta. \quad (3.7)$$

**Corollary 3.1.** Let $u = M(q)$, $v = M(q^2)$, $w = M(q^3)$, $x = M(q^4)$, $y = M(q^6)$ and $z = M(q^8)$. Then

(i) $y^4u^4 - w^4v^4 - 4y^3v^3wu + w^3u^3yv = 0,$

(ii) $(x^2 + 2z^2)^4(u^8 - 16v^8) - u^8(x^2 - 2z^2)^4 = 0,$

and

(iii) $[u^8 - 16v^8][w^8 - 16y^8] = [u^2w^2 - 4v^2y^2]^4. \quad (3.10)$

**Proof of (i):** When $\beta$ has degree 3 over $\alpha$, we have [7, p. 231, Entry 5 (xiii)]

$$\left(\frac{\beta}{\alpha}\right)^{1/4} - \left(\frac{\alpha}{\beta}\right)^{1/4} = 2((\alpha\beta)^{1/8} - (\alpha\beta)^{-1/8}). \quad (3.11)$$

Using (3.7) with $n = 3$ and (3.2), it can be seen that (3.11) is equivalent to

$$\left(\frac{yu}{wv}\right)^2 - \left(\frac{wv}{yu}\right)^2 = 4 \frac{vy}{wu} - \frac{wu}{vy}. \quad (3.12)$$

We obtain (3.8) upon simplifying (3.12).

**Proof of (ii):** When $\beta$ has degree 4 over $\alpha$, we have [7, Eq. (24.22), p.215]

$$\sqrt{\beta} = \left(\frac{1 - (1 - \alpha)^{1/4}}{1 + (1 - \alpha)^{1/4}}\right)^2.$$

Replacing $\alpha$ by $(1 - \beta)$ and $\beta$ by $(1 - \alpha)$ we obtain [7, Entry 24 (v),p.216]

$$\sqrt{1 - \alpha} = \left(\frac{1 - \beta^{1/4}}{1 + \beta^{1/4}}\right)^2. \quad (3.13)$$
Using (3.7) with \(n = 4\), and (3.2) it can be seen that, (3.13) is equivalent to
\[
\left(1 - \frac{16v^8}{u^8}\right)^{1/2} = \left(\frac{x^2 - 2z^2}{x^2 + 2z^2}\right)^2.
\]
(3.14)

Squaring both sides of (3.14) and then simplifying, we obtain (3.9).

**Proof of (iii):** When \(\beta\) has degree 3 over \(\alpha\), we have [7, Entry 5 (ii), p. 230]
\[
(\alpha\beta)^{1/4} + (1 - \alpha)^{1/4}(1 - \beta)^{1/4} = 1.
\]
(3.15)

Using (3.7) with \(n = 3\), and (3.2) it can be seen that (3.15) is equivalent to
\[
\left(1 - \frac{16v^8}{u^8}\right)^{1/4} \left(1 - \frac{16y^8}{w^8}\right)^{1/4} = \left(1 - \frac{4v^2y^2}{w^2w^2}\right).
\]
(3.16)

Taking fourth power of both sides of (3.16) and then simplifying, we obtain (3.10).

**4. Some Evaluations of \(M(q)\)**

We now establish some evaluations of \(M(q)\) using Theorem 3.1. Let \(q_n := e^{-\pi\sqrt{n}}\) and let \(\alpha_n\) denote the corresponding value of \(\alpha\) in (3.1). Then by Theorem 3.1, we have
\[
\frac{M(e^{-2\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}})} = \frac{1}{\sqrt{2}}\alpha_n^{1/8}.
\]
(4.1)

It is known [7, p. 97] that \(\alpha_1 = \frac{1}{2}\), \(\alpha_2 = (\sqrt{2} - 1)^2\) and \(\alpha_4 = (\sqrt{2} - 1)^4\). Hence, using (4.1), we obtain
\[
\frac{M(e^{-2\pi})}{M(e^{-\pi})} = \left(\frac{1}{2}\right)^{5/8},
\]
(4.2)

\[
\frac{M(e^{-2\pi\sqrt{2}})}{M(e^{-\pi\sqrt{2}})} = \frac{1}{\sqrt{2}}(\sqrt{2} - 1)^{1/4}
\]
(4.3)

and
\[
\frac{M(e^{-4\pi})}{M(e^{-2\pi})} = \frac{1}{\sqrt{2}}(\sqrt{2} - 1)^{1/2}.
\]
(4.4)
Ramanujan has recorded many modular equations in his notebooks, which are very useful in the computation of class invariants and the values of theta-functions. In the literature not much attention has been paid to find the values of $\psi(q)$ and $\varphi(q)$. However, Ramanujan has recorded several values of $\varphi(q)$ and $\psi(q)$ in his notebooks. For example

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)}$$

and

$$\psi(e^{-\pi}) = 2^{-5/8}e^{\pi/8} \frac{\pi^{1/4}}{\Gamma(3/4)}.$$  \hspace{1cm} (4.6)

Using (4.6), (3.4) and (4.2), we deduce that

$$M(e^{-2\pi}) = 2^{-5/4} \frac{\pi^{1/4}}{\Gamma(3/4)}.$$  \hspace{1cm} (4.7)

Using (4.7) in (4.4), we obtain

$$M(e^{-4\pi}) = 2^{-7/4}(\sqrt{2} - 1)^{1/2} \frac{\pi^{1/4}}{\Gamma(3/4)}.$$  

The Ramanujan-Weber class invariants are defined by

$$G_n := 2^{-1/4} q_n^{-1/24} (-q_n; q_n^2)_{\infty}$$

and

$$g_n := 2^{-1/4} q_n^{-1/24} (q_n; q_n^2)_{\infty}$$

where $q_n := e^{-\pi \sqrt{n}}$. Chan and Huang [13] have derived some explicit formulas for evaluating $H(e^{-\pi \sqrt{n}/2})$ in terms of Ramanujan-Weber class invariants. On the same lines one can show that

$$\frac{M(e^{-2\pi \sqrt{n}})}{M(e^{-\pi \sqrt{n}})} = \frac{1}{\sqrt{2} (\sqrt{p(p+1)} + \sqrt{p(p-1)})^{1/4}} = \frac{1}{\sqrt{2} (p_1 + \sqrt{p_1^2 + 1})^{1/4}}$$

where $p = G_n^{12}$ and $p_1 = g_n^{12}$. 

Finally, we remark that $M(q)$ and $H(q)$ are related by the equation

$$H^{-1}(\sqrt{q}) - H(\sqrt{q}) = \frac{M^2(q)}{M^2(q^2)}.$$

Changing $q$ to $q^2$ in the above, we obtain

$$M^2(q^4)H^2(q) + M^2(q^2)H(q) - M^2(q^4) = 0.$$

From this equation we can compute $H(q)$, using the known values of $M(q^2)$ and $M(q^4)$. On the other hand we can also compute $M(q^4)$, using the known values of $H(q)$ and $M(q^2)$.

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References


