Relationship between certain classes of life distributions and some stochastic orderings

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Abstract. We consider distribution functions (dfs) in NBU, NBUE, NBUC and NBUT classes of life distributions and study the stochastic orderings of the associated random variables (rvs), their equilibrium and residual rvs at fixed and at random times. These results offer more insight into the structure of these classes.

1. Introduction

We start with a few definitions which may be found in Barlow and Proschan [1], Shaked and Shanthikumar [2], Szekli [3] and Yue and Cao [4]. Let $X$ and $Y$ be two nonnegative independent random variables (rvs) with respective distribution functions (dfs) $F$ and $G$, survival functions (sfs) $F$ and $G$. For $t \geq 0$, let $X_t$ denote the excess or residual or remaining life rv at time $t$ with df $F_t(x) = P(X_t \leq x) = 0, x \leq 0$, and $F_t(x) = F(t+x) - F(t), x > 0, t > 0$.

A few definitions:

1. $X$ is said to be stochastically smaller than $Y$, denoted by $X \leq_{st} Y$, if $F(t) \leq G(t), t \geq 0$.
2. $X$ is smaller than $Y$ in convex order, denoted by $X \leq_{c} Y$, if $\int_{0}^{\infty} F(u)du \leq \int_{0}^{\infty} G(u)du, t \geq 0$.
3. $X$ is said to be smaller than $Y$ in the total time on test transform (ttt) order, denoted by $X \leq_{ttt} Y$, if $\int_{0}^{\infty} F^{-1}(p)F(u)du \leq \int_{0}^{\infty} G^{-1}(p)G(u)du, p \in (0, 1)$, where $F^{-1}(p)$ denotes the $p$-th quantile of $F$.
4. $X$ is said to be smaller than $Y$ with respect to the Laplace-Stieltjes transform order, denoted by $X <_{L} Y$, if $E(e^{-sX}) \geq E(e^{-sY}), s \geq 0 \iff \int_{0}^{\infty} e^{-su}dF(u) \geq \int_{0}^{\infty} e^{-su}dG(u), s \geq 0$

5. $F$ is said to be New Better than used or NBU if $\mathcal{F}(s + t) \leq \mathcal{F}(s)\mathcal{F}(t), s \geq 0, t \geq 0$.

6. $F$ is said to be New better than used in expectation or NBUE if
   (a) $F$ has finite mean $\mu_F$,

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Theorem 2.1. Yue and Cao [4] prove that

If

Let, then:

Proof. (NWUL) iff

\[ \int_0^\infty e^{-su}F(t+u)du \leq F(t) \int_0^\infty e^{-su}F(u)du, \ t \geq 0. \]

8. \( F \) is said to be New Better than used in convex order or NBUC if \( X \leq_{\text{icx}} X, \ t \geq 0 \), where \( \text{Y} \leq_{\text{icx}} Z \) means that \( Y \) is bounded by \( Z \) in increasing convex order. Equivalently, \( F \) is NBUC iff

\[ \int_x^\infty F(t+u)du \leq F(t) \int_x^\infty F(u)du, \ x \geq 0, \ t \geq 0. \]

9. \( F \) is said to be New Better than used in total time on test order or NBUT if

\[ \int_0^{F^{-1}(p)} F(t+u)du \leq F(t) \int_0^{F^{-1}(p)} F(u)du, \ 0 < p < 1, \ t \geq 0. \]

The dual stochastic orders/classes are defined by reversing the inequalities.

2. Main Results

Let \( XY \) denote the residual life rv at random time \( Y \). Then \( XY \) has sf

\[ F_{XY}(t) = P(X-Y > t|X > Y) = 0, \ t \leq 0, \]

and

\[ F_{XY}(t) = \frac{\int_0^\infty F(t+y)G(y)dy}{\int_0^\infty F(y)G(y)dy}, \ t > 0. \]

Yue and Cao [4] prove that \( F \) is NBUL (NWUL) iff \( XY \leq_{\text{LT}} (\geq_{\text{LT}}) X \) for any rv \( Y \) and also that \( F \) is NBUL (NWUL) iff \( XY \leq_{\text{st}} (\geq_{\text{st}}) X \) for any exponential rv \( Y \). We look at some ordering relations satisfied by these rvs.

Theorem 2.1. \( F \) is NBU (NWU) iff \( XY \leq_{\text{st}} (\geq_{\text{st}}) X \) where \( Y \) is any nonnegative rv.

Proof. Let \( Y \) be an arbitrary nonnegative rv with df \( G \). Then

\[ F \text{ is NBU} \implies \frac{F(x+y)}{F(y)} \leq \frac{F(x)}{F(y)}, \ x \geq 0, \ y \geq 0 \]

\[ \implies \int_0^\infty \frac{F(x+y)G(y)}{F(y)G(y)}dy \leq \int_0^\infty \frac{F(x)G(y)}{F(y)G(y)}dy, \ x \geq 0 \]

\[ \implies \frac{\int_0^\infty F(x+y)G(y)dy}{\int_0^\infty F(y)G(y)dy} \leq \frac{F(x)}{F(y)}, \ x \geq 0 \]

\[ \implies F_{XY}(x) \leq F(x), \ x \geq 0 \implies XY \leq_{\text{st}} X. \]

Conversely, if \( XY \leq_{\text{st}} X \) for all nonnegative rvs \( Y \), taking \( Y \equiv t, \ t \geq 0 \), we get

\[ X_t \leq_{\text{st}} X \implies F_t(x) \leq F(x), \ x \geq 0 \implies \frac{F(t+x)}{F(t)} \leq F(x), \ x \geq 0 \implies F \text{ is NBU}. \]

Hence the proof. \( \square \)

The following result gives relationship between some of these stochastic orders.

Theorem 2.2. If \( XY \leq_{\text{st}} X \), then: (i) \( XY \leq_{\text{c}} X \), (ii) \( XY \leq_{\text{LT}} X \), and (iii) \( XY \leq_{\text{st}} X \).
Proof.

(i) $X_Y \leq_t X \Rightarrow F_{X_Y}(x) \leq F(x)$, $x \geq 0 \Rightarrow \int_t^\infty F_{X_Y}(x)dx \leq \int_t^\infty F(x)dx$, $t \geq 0 \Rightarrow X_Y \leq_c X$.

(ii) $X_Y \leq_{st} X \Rightarrow e^{-sx}F_{X_Y}(x) \leq e^{-sx}F(x)$, $x \geq 0$,
\[ \Rightarrow \int_0^\infty e^{-sx}F_{X_Y}(x)dx \leq \int_0^\infty e^{-sx}F(x)dx, \quad s \geq 0 \]
\[ \Rightarrow X_Y \leq_L X. \]

(iii) $X_Y \leq_{st} X \Rightarrow \int_0^{F_{X_Y}^{-1}(p)} F_{X_Y}(x)dx \leq \int_0^{F_{X_Y}^{-1}(p)} F(x)dx, \quad 0 < p < 1$
\[ \Rightarrow \int_0^{F_{X_Y}^{-1}(p)} F_{X_Y}(x)dx \leq \int_0^{F_{X_Y}^{-1}(p)} F(x)dx \text{ since } F_{X_Y}^{-1}(p) \leq F^{-1}(p), 0 < p < 1, \]
\[ \Rightarrow X_Y \leq_{tt} X. \]

\[ \square \]

Theorem 2.3. If $F$ is NBU, then: (i) $X_Y \leq_{st} X$, (ii) $X_Y \leq_c X$, (iii) $X_Y \leq_L X$, and (iv) $X_Y \leq_{tt} X$.

Proof. (i) follows from Theorem 2.1 and the rest follow from Theorem 2.1 and Theorem 2.2. \[ \square \]

Theorem 2.4. $F$ is NBUC(NWUC) iff $X_Y \leq_c (\geq_c)X$ where $Y$ is any nonnegative rv.

Proof.

$F$ is NBUC \[ \Rightarrow \int_x^\infty F(u+y)du \leq F(y)\int_x^\infty F(u)du, \quad x \geq 0, \]
\[ \Rightarrow \int_x^\infty \int_x^\infty F(u+y)dudG(y) \leq \int_x^\infty F(y)dG(y)\int_x^\infty F(u)du, \quad x \geq 0, \]
\[ \Rightarrow \int_x^\infty \int_0^\infty F(u+y)dG(y)du \leq \int_x^\infty F(u)du, \quad x \geq 0, \]
\[ \Rightarrow \int_x^\infty F_{X_Y}(x)dx \leq \int_x^\infty F(u)du, \quad x \geq 0, \Rightarrow X_Y \leq_c X. \]

Conversely, if $X_Y \leq_c X$ holds for all nonnegative rvs $Y$, taking $Y \equiv t$, $t \geq 0$,
\[ X_t \leq_c X \Leftrightarrow \int_x^\infty F_t(u)du \leq \int_x^\infty F(u)du, \quad x \geq 0, \]
\[ \Leftrightarrow \int_x^\infty \frac{F(t+u)}{F(t)}du \leq \int_x^\infty F(u)du, \quad x \geq 0, \]
\[ \Leftrightarrow \int_x^\infty F(t+u)du \leq F(t)\int_x^\infty F(u)du, \quad x \geq 0, \Leftrightarrow F$ is NBUC.

Hence the proof. \[ \square \]

Theorem 2.5. (i) $X_t \leq_{st} X \Leftrightarrow F$ is NBU.

(ii) $X_t \leq_c X \Leftrightarrow F$ is NBUC.

(iii) $X_t \leq_L X \Leftrightarrow F$ is NBUL.

(iv) $X_t \leq_{tt} X \Leftrightarrow F$ is NBUT.
Proof.

(i) $X_t \leq_{st} X \iff F_t(x) \leq F(x), \; x \geq 0, \; t \geq 0$
\[ \iff \frac{F(t+x)}{F(t)} \leq F(x), \; x \geq 0, \; t \geq 0 \]
\[ \iff F(t+x) \leq F(t)F(x), \; x \geq 0, \; t \geq 0 \iff F \text{ is NBU}.
\]

(ii) $X_t \leq_c X \iff \int_x^\infty F_t(u)du \leq \int_x^\infty F(u)du, \; x \geq 0, \; t \geq 0$
\[ \iff \int_x^\infty \frac{F(t+u)}{F(t)}du \leq \int_x^\infty F(u)du, \; x \geq 0, t \geq 0, \]
\[ \iff \int_x^\infty F(t+u)du \leq F(t) \int_x^\infty F(u)du, \; x \geq 0, t \geq 0, \iff F \text{ is NBUC}.
\]

(iii) $X_t \leq_L X \iff \int_0^\infty e^{-sx}F_t(x)dx \leq \int_0^\infty e^{-sx}F(x)dx, \; s \geq 0$
\[ \iff \int_0^\infty e^{-sx}\frac{F(t+x)}{F(t)}dx \leq \int_0^\infty e^{-sx}F(x)dx, \; s \geq 0 \]
\[ \iff \int_0^\infty e^{-sx}F(t+x)dx \leq F(t) \int_0^\infty e^{-sx}F(x)dx, \; s \geq 0 \iff F \text{ is NBUL}.
\]

(iv) $X_t \leq_{tt} X, \; t \geq 0 \iff \int_0^{F^{-1}(p)} F_t(x)dx \leq \int_0^{F^{-1}(p)} F(x)dx, \; t \geq 0, \; 0 < p < 1$
\[ \iff \int_0^{F^{-1}(p)} \frac{F(t+x)}{F(t)}dx \leq \int_0^{F^{-1}(p)} F(x)dx, \; t \geq 0, \; 0 < p < 1, \]
\[ \iff \int_0^{F^{-1}(p)} F(t+x)dx \leq F(t) \int_0^{F^{-1}(p)} F(x)dx, \; t \geq 0, \; 0 < p < 1, \]
\[ \iff F \text{ is NBUT}.
\]

The next few results are concerned about equilibrium distributions. Associated with the rv $X$ with df $F$ and mean $\mu_F$, let $X_1$ denote the equilibrium rv with df $F_1(x) = \frac{1}{\mu_F} \int_0^x F(t)dt, \; x \geq 0$, and $F_1(x) = 0$ otherwise. Then $F_1(x) = \frac{1}{\mu_F} \int_x^\infty F(t)dt, \; x \geq 0$.

**Theorem 2.6.** If $F$ is NBUE, then: (i) $X_1 \leq_c X$, (ii) $X_1 \leq_{tt} X$, (iii) $X_1 \leq_L X$.

**Proof.**

(i) $F$ is NBUE \[ \iff \int_0^\infty F(x)dx \leq F(t), \; t \geq 0, \]
\[ \iff F_1(t) \leq F(t), \; t \geq 0, \]
\[ \Rightarrow \int_x^\infty F_1(t)dt \leq \int_x^\infty F(t)dt, \; x \geq 0, \Rightarrow X_1 \leq_c X.
\]
Theorem 2.9. \( F \) is NBUE \( \Rightarrow \) \( \int_{0}^{F^{-1}(p)} F_1(t)dt \leq \int_{0}^{F^{-1}(p)} F(t)dt, \ 0 < p < 1, \text{ using (1)} \)
\[ \Rightarrow \int_{0}^{F^{-1}(p)} F_1(t)dt \leq \int_{0}^{F^{-1}(p)} F(t)dt, \text{ since } F_1^{-1}(p) \leq F^{-1}(p) \]
\[ \Rightarrow X_1 \leq_{st} X. \]

(iii) \( F \) is NBUE \( \Rightarrow \) \( e^{-st}F_1(t) \leq e^{-st}F(t), \ s \geq 0, \ t \geq 0, \)
\[ \Rightarrow \int_{0}^{\infty} e^{-st}F_1(t)dt \leq \int_{0}^{\infty} e^{-st}F(t)dt, s \geq 0, \Rightarrow X_1 \leq_{\leq} X. \]

\[ \square \]

Theorem 2.7. \( X_1 \leq_{st} X \Leftrightarrow F \) is NBUE.

Proof.
\[ X_1 \leq_{st} X \Leftrightarrow F_1(t) \leq F(t) \Leftrightarrow \frac{1}{\mu} \int_{t}^{\infty} F(x)dx \leq F(t), \ t \geq 0, \Leftrightarrow F \) is NBUE.

\[ \square \]

We next obtain a few results using the equilibrium distribution of the equilibrium distribution. These results show that the properties hold even if we iterate formation of equilibrium distributions. Let \( X_2 \) denote the equilibrium rv of \( X_1 \), with df \( F_2(x) = \frac{1}{\mu(F_1)} \int_{0}^{x} F_1(t)dt, \ x \geq 0, \) and \( F_2(x) = 0 \) otherwise, where \( \mu(F_1) \) is the mean of \( X_1 \) and \( F_1 \) is the df of \( X_1 \). Then \( F_2(x) = \frac{1}{\mu(F_1)} \int_{x}^{\infty} F_1(t)dt, \ x \geq 0. \)

Theorem 2.8. \( X_2 \leq_{st} X_1 \Leftrightarrow F_1 \) is NBUE.

Proof.
\[ X_2 \leq_{st} X_1 \Leftrightarrow F_2(t) \leq F_1(t) \Leftrightarrow \frac{1}{\mu(F_1)} \int_{t}^{\infty} F_1(x)dx \leq F_1(t), \ t \geq 0, \Leftrightarrow F_1 \) is NBUE.

\[ \square \]

Theorem 2.9. Let \( F_1 \) be NBUE. Then: (i) \( X_2 \leq_{c} X_1 \), (ii) \( X_2 \leq_{tt} X_1 \), (iii) \( X_2 \leq_{\leq} X_1 \).

Proof.

(i) \( F_1 \) is NBUE \( \Leftrightarrow \) \( \frac{1}{\mu(F_1)} \int_{t}^{\infty} F_1(x)dx \leq F_1(t), \ t \geq 0, \)
\[ \Leftrightarrow F_2(t) \leq F_1(t), \ t \geq 0, \]
\[ \Rightarrow \int_{x}^{\infty} F_2(t)dt \leq \int_{x}^{\infty} F_1(t)dt, \ x \geq 0, \]
\[ \Rightarrow X_2 \leq_{c} X_1. \]

(ii) \( F_1 \) is NBUE \( \Rightarrow \) \( \int_{0}^{F^{-1}(p)} F_2(t)dt \leq \int_{0}^{F^{-1}(p)} F_1(t)dt, \ 0 < p < 1, \text{ using (2)} \)
\[ \Rightarrow \int_{0}^{F^{-1}(p)} F_2(t)dt \leq \int_{0}^{F^{-1}(p)} F_1(t)dt, \ 0 < p < 1, \text{ since } F_2^{-1}(p) \leq F_1^{-1}(p) \]
\[ \Rightarrow X_2 \leq_{tt} X_1. \]
(iii) $F_1$ is NBUE $\Rightarrow e^{-st}F_2(t) \leq e^{-st}F_1(t)$, $t \geq 0$, $s \geq 0$, using (2)

$\Rightarrow \int_0^\infty e^{-st}F_2(t)dt \leq \int_0^\infty e^{-st}F_1(t)dt$, $s \geq 0$,

$\Rightarrow X_2 \leq_{st} X_1$.

The next property is a consequence of the NBU property.

**Theorem 2.10.** If $X \sim F$ is NBU, then $X_n \leq_{st} X$, $n = 1, 2, 3$, where $X_1 \sim F_1$ is the equilibrium rv derived from $X$, $X_n \sim F_n$ is the equilibrium rv derived from $X_{n-1}$, $n = 2, 3$.

**Remark 2.11.** The statement appears to be true for other values of $n \geq 4$, but we have not been able to prove it using induction.

**Proof.** We denote the mean of $F$ by $\mu_F$ and that of $F_k$, $k = 1, 2, 3$.

We have $X_1 \sim F_1(t) \iff F_1(t) = 1/\mu_F \int_t^\infty f(u)du, t \geq 0$

$\iff F_1(t) = 1/\mu_F \int_0^\infty (s+t)ds, t \geq 0$,

$\Rightarrow F_1(t) \leq F(t), t \geq 0$, if $X$ is NBU.

We have $X_2 \sim F_2(t) \iff F_2(t) = 1/\mu_{F_1} \int_t^\infty f(u)du, t \geq 0$,

$\iff F_2(t) = 1/\mu_{F_2} \int_t^\infty f(u)du, t \geq 0$,

$\Rightarrow F_2(t) = 2/\mu_{F_2} \int_t^\infty 1/\mu_F \int_u^\infty F(x)dxdu, t \geq 0$,

$\Rightarrow F_2(t) = 2/\mu_{F_2} \int_t^\infty \int_u^\infty duF(x)dx, t \geq 0$,

$\Rightarrow F_2(t) = 2/\mu_{F_2} \int_t^\infty (x-t)F(x)dx, t \geq 0$,

$\Rightarrow F_2(t) = 2/\mu_{F_2} \int_t^\infty uF(t+u)du, t \geq 0$,

$\Rightarrow F_2(t) \leq 2/\mu_{F_2} F(t+u)du, t \geq 0$, if $F$ is NBU,

$\Rightarrow F_2(t) \leq 2/\mu_{F_2} F(t)du, t \geq 0$,

$\Rightarrow F_2(t) \leq 2F(t)/\mu_{F_2} = F(t), t \geq 0 \Rightarrow X_2 \leq_{st} X$.

The proof for the case $n = 3$ is similar and is omitted. $\square$
References